

On Some Properties of Space Inverses of Stochastic Flows

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Abstract

We derive moment estimates and a strong limit theorem for space inverses of stochastic flows generated by jump SDEs with adapted coefficients in weighted Hölder norms using the Sobolev embedding theorem and the change of variable formula. As an application of some basic properties of flows of continuous SDEs, we derive the existence and uniqueness of classical solutions of linear parabolic second order SPDEs by partitioning the time interval and passing to the limit. The methods we use allow us to improve on previously known results in the continuous case and to derive new ones in the jump case.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual conditions of right-continuity and completeness. Let $(w_t^\varrho)_{\varrho \geq 1}, t \geq 0, \varrho \in \mathbf{N}$, be a sequence of independent one-dimensional \mathbf{F} -adapted Wiener processes. For a (Z, \mathcal{Z}, π) is a sigma-finite mea-

sure space, we let $p(dt, dz)$ be an \mathbf{F} -adapted Poisson random measure on $(\mathbf{R}_+ \times Z, \mathcal{B}(\mathbf{R}_+) \otimes \mathcal{Z})$ with intensity measure $\pi(dz)dt$ and denote by $q(dt, dz) = p(dt, dz) - \pi(dz)dt$ the compensated Poisson random measure. For each real number $T > 0$, we let \mathcal{R}_T and \mathcal{P}_T be the \mathbf{F} -progressive and \mathbf{F} -predictable sigma-algebra on $\Omega \times [0, T]$, respectively.

Fix a real number $T > 0$ and an integer $d \geq 1$. For each stopping time $\tau \leq T$, consider the stochastic flow $X_t = X_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$, generated by the stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= b_t(X_t)dt + \sigma_t^{\varrho}(X_t)dw_t^{\varrho} + \int_Z H_t(X_{t-}, z)q(dt, dz), \quad \tau < t \leq T, \\ X_t &= x, \quad t \leq \tau, \end{aligned} \tag{1.1}$$

where $b_t(x) = (b_t^i(\omega, x))_{1 \leq i \leq d}$ and $\sigma_t(x) = (\sigma_t^{i\varrho}(\omega, x))_{1 \leq i \leq d, \varrho \geq 1}$ are $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable random fields defined on $\Omega \times [0, T] \times \mathbf{R}^d$ and $H_t(x, z) = (H_t^i(\omega, x, z))_{1 \leq i \leq d}$ is a $\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{Z}$ -measurable random fields defined on $\Omega \times [0, T] \times \mathbf{R}^d \times Z$. The summation convention with respect to the repeated index $\varrho \in \mathbf{N}$ is used here and below. In this paper, under natural regularity assumptions on the coefficients b , σ , and H , we provide a simple and direct derivation of moment estimates of the space inverse of the flow, denoted $X_t^{-1}(\tau, x)$, in weighted Hölder norms by applying the Sobolev embedding theorem and the change of variable formula. Using a similar method, we establish a strong limit theorem in weighted Hölder norms for a sequence of flows $X_t^{(n)}(\tau, x)$ and their inverses $X_t^{(n);-1}(\tau, x)$ corresponding to a sequence of coefficients $(b^{(n)}, \sigma^{(n)}, H^{(n)})$ converging in an appropriate sense. Furthermore, as an application of the diffeomorphism property of flow, we give a direct derivation of the linear second order degenerate stochastic partial differential equation (SPDE) governing the inverse flow $X_t^{-1}(\tau, x)$ when $H \equiv 0$. Specifically, for each $\tau \leq T$, consider the stochastic flow $Y_t = Y_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$, generated by the SDE

$$\begin{aligned} dY_t &= b_t(Y_t)dt + \sigma_t^{\varrho}(Y_t)dw_t^{\varrho}, \quad \tau < t \leq T, \\ Y_t &= x, \quad t \leq \tau. \end{aligned}$$

Assume that b and σ have linear growth, bounded first and second derivatives, and that the second derivatives of b and σ are α -Hölder for some $\alpha > 0$. By partitioning the time interval and using Taylor's theorem, the Sobolev embedding theorem, and some basic properties of the flow and its inverse, we show that $u_t(x) = u_t(\tau, x) := Y_t^{-1}(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$ is the unique classical solution of the SPDE given by

$$\begin{aligned} du_t(x) &= \left(\frac{1}{2} \sigma_t^{i\varrho}(x) \sigma_t^{j\varrho}(x) \partial_{ij} u_t(x) - \hat{b}_t^i(x) \partial_i u_t(x) \right) dt - \sigma_t^{i\varrho}(x) \partial_i u_t(x) dw_t^{\varrho}, \quad \tau < t \leq T, \\ u_t(x) &= x, \quad t \leq \tau, \end{aligned} \tag{1.2}$$

where

$$\hat{b}_t^i(x) = b_t^i(x) - \sigma_t^{j\varrho}(x) \partial_j \sigma_t^{i\varrho}(x).$$

In [LM14], we use all of the properties of the flow $X_t(\tau, x)$ that are established in this work in order to derive the existence and uniqueness of classical solutions of linear parabolic

stochastic integro-differential equations (SIDEs).

One of the earliest works to investigate the homeomorphism property of flows of SDEs with jumps is by P. Meyer in [Mey81]. In [Mik83], R. Mikulevičius extended the properties found in [Mey81] to SDEs driven by arbitrary continuous martingales and random measures. Many other authors have since expanded upon the work in [Mey81], see for example [FK85, Kun04, MB07, QZ08, Zha13, Pri14] and references therein. In [Kun04, Kun86], H. Kunita studied the diffeomorphism property of the flow $X_t(s, x)$, $(s, t, x) \in [0, T]^2 \times \mathbf{R}^d$, and in the setting of deterministic coefficients, he showed that for each fixed t , the inverse flow $X_t^{-1}(s, x)$, $(s, x) \in [t, T] \times \mathbf{R}^d$, solves a backward SDE. By estimating the associated backward SDE, one can obtain moment estimates and a strong limit theorem for the inverse flow in essentially the same way that moment estimates are obtained for the direct flow (see, e.g. [Kun86]). However, this method of deriving moment estimates and a strong limit theorem for the inverse flow uses a time reversal, and thus requires that the coefficients are deterministic. In the case $H \equiv 0$, numerous authors have investigated properties of the inverse flow with random coefficients. In Chapter 2 of [Bis81], Lemma 2.1 and 2.2. of [OP89], and Section 6.1 and 6.2 of [Kun96], the authors derive properties of $Y_t^{-1}(\tau, x)$ (e.g. moment estimates, strong limit theorem, and the fact that it solves (1.2)) by first showing that it solves the Stratonovich form SDE for $Z_t = Z_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$, given by

$$\begin{aligned} dZ_t(x) &= -U_t(Z_t(x))b_t(x)dt - U_t(Z_t(x))\sigma_t^\circ(x) \circ dw_t^\circ, \quad \tau < t \leq T, \\ Z_0(x) &= x, \quad \tau < t, \end{aligned} \tag{1.3}$$

where $U_t(x) = U_t(\tau, x) = [\nabla Y_t(\tau, x)]^{-1}$. In order to obtain a strong solution to (1.3), the authors impose conditions on the coefficients that guarantee $\nabla U_t(x)$ is locally-Lipschitz in x . In the degenerate setting, the third derivative of b_t and σ_t need to be α -Hölder for some $\alpha > 0$ to obtain that $\nabla U_t(x)$ is locally-Lipschitz in x . However, for some reason, the authors assume more regularity than this. In this paper, we derive properties of the inverse flow under those assumptions which guarantee that $Y_t(\tau, x)$ is a C_{loc}^β -diffeomorphism (and with $\beta > 1$).

Classical solutions of (1.2) have been constructed in [Bis81, Kun96] by directly showing that $Y_t^{-1}(\tau, x)$ solves (1.3). As we have mentioned above, this approach requires the third derivatives of b_t and σ_t to be α -Hölder for some $\alpha > 0$. Yet another approach to deriving existence of classical solutions of (1.2) is using the method of time reversal (see, e.g. [Kun96, DPT98]). While this method only requires that the second derivatives of b_t and σ_t are α -Hölder for some $\alpha > 0$, it does impose that the coefficients are deterministic. In [KR82], N.V. Krylov and B.L. Rozvskii derived the existence and uniqueness of generalized solutions of degenerate second order linear parabolic SPDEs in Sobolev spaces using variational approach of SPDEs and the method of vanishing viscosity (see, also, [GGK14] and Ch. 4, Sec. 2, Theorem 1 in [Roz90]). Thus, by appealing to the Sobolev embedding theorem, this theory can be used to obtain classical solutions of degenerate linear SPDEs. Proposition 1 of Ch. 5, Sec. 2, in [Roz90] shows that if σ is uniformly bounded and four-times continuously differentiable in x with uniformly bounded derivatives and b is uniformly bounded and three-times continuously differentiable with uniformly bounded derivatives, then there exists a classical solution of (1.2) and $u_t(x) = Y_t^{-1}(x)$. This is more regularity than we require.

This paper is organized as follows. In Section 2, we state our notation and the main results. Section 3 is devoted to the proof of the properties of the stochastic flow $X_t(\tau, x)$ and Section 4 to the proof that $Y_t^{-1}(\tau, x)$ is the unique classical solution of (1.2). In Section 5, the appendix, auxiliary facts that are used throughout the paper are discussed.

2 Outline of main results

For each integer $n \geq 1$, let \mathbf{R}^n be the n -dimensional Euclidean space and for each $x \in \mathbf{R}^n$, denote by $|x|$ the Euclidean norm of x . Let \mathbf{R}_+ denote the set of non-negative real-numbers. Let \mathbf{N} be the set of natural numbers. Elements of \mathbf{R}^d are understood as column vectors and elements of \mathbf{R}^{2d} are understood as matrices of dimension $d \times d$. We denote the transpose of an element $x \in \mathbf{R}^d$ by x^* . The norm of an element x of $\ell_2(\mathbf{R}^d)$ (resp. $\ell_2(\mathbf{R}^{2d})$), the space of square-summable \mathbf{R}^d -valued (resp. \mathbf{R}^{2d} -valued) sequences, is also denoted by $|x|$. For a topological space (X, \mathcal{X}) we denote the Borel sigma-field on X by $\mathcal{B}(X)$.

For each $i \in \{1, \dots, d_1\}$, let $\partial_i = \frac{\partial}{\partial x_i}$ be the spatial derivative operator with respect to x_i and write $\partial_{ij} = \partial_i \partial_j$ for each $i, j \in \{1, \dots, d_1\}$. For a once differentiable function $f = (f^1, \dots, f^{d_1}) : \mathbf{R}^{d_1} \rightarrow \mathbf{R}^{d_1}$, we denote the gradient of f by $\nabla f = (\partial_j f^i)_{1 \leq i, j \leq d_1}$. Similarly, for a once differentiable function $f = (f^{1\varrho}, \dots, f^{d\varrho})_{\varrho \geq 1} : \mathbf{R}^{d_1} \rightarrow \ell_2(\mathbf{R}^{d_1})$, we denote the gradient of f by $\nabla f = (\partial_j f^{i\varrho})_{1 \leq i, j \leq d_1, \varrho \geq 1}$ and understand it as a function from \mathbf{R}^{d_1} to $\ell_2(\mathbf{R}^{2d_1})$. For a multi-index $\gamma = (\gamma_1, \dots, \gamma_d) \in \{0, 1, 2, \dots\}^{d_1}$ of length $|\gamma| := \gamma_1 + \dots + \gamma_d$, denote by ∂^γ the operator $\partial^\gamma = \partial_1^{\gamma_1} \dots \partial_d^{\gamma_d}$, where ∂_i^0 is the identity operator for all $i \in \{1, \dots, d_1\}$. For each integer $d \geq 1$, we denote by $C_c^\infty(\mathbf{R}^{d_1}; \mathbf{R}^d)$ the space of infinitely differentiable functions with compact support in \mathbf{R}^d .

For a Banach space V with norm $|\cdot|_V$, domain Q of \mathbf{R}^d , and continuous function $f : Q \rightarrow V$, we define

$$|f|_{0;Q;V} = \sup_{x \in Q} |f(x)|$$

and

$$[f]_{\beta;Q;V} = \sup_{x, y \in Q, x \neq y} \frac{|f(x) - f(y)|_V}{|x - y|_V^\beta}, \quad \beta \in (0, 1].$$

For each real number $\beta \in \mathbf{R}$, we write $\beta = [\beta]^- + \{\beta\}^+$, and $\{\beta\}^+ \in (0, 1]$. For a Banach space V with norm $|\cdot|_V$, real number $\beta > 0$, and domain Q of \mathbf{R}^d , we denote by $C^\beta(Q; V)$ the Banach space of all bounded continuous functions $f : Q \rightarrow V$ having finite norm

$$|f|_{\beta;Q;V} := \sum_{|\gamma| \leq [\beta]^-} |\partial^\gamma f|_{0;Q;V} + \sum_{|\gamma| = [\beta]^-} [\partial^\gamma f]_{\{\beta\}^+;Q;V}.$$

When $Q = \mathbf{R}^d$ and $V = \mathbf{R}^n$ or $V = \ell_2(\mathbf{R}^n)$ for any integer $n \geq 1$, we drop the subscripts Q and V from the norm $|\cdot|_{\beta;Q;V}$ and write $|\cdot|_\beta$. For a Banach space V and for each $\beta > 0$, denote by $C_{loc}^\beta(\mathbf{R}^d; V)$ the Fréchet space of continuous functions $f : \mathbf{R}^d \rightarrow V$ satisfying $f \in C^\beta(Q; V)$ for all bounded domains $Q \subset \mathbf{R}^d$. We call a function $f : \mathbf{R}^d \rightarrow \mathbf{R}^d$ a $C_{loc}^\beta(\mathbf{R}^d; \mathbf{R}^d)$ -diffeomorphism if f is a homeomorphism and both f and its inverse f^{-1} are

in $C_{loc}^\beta(\mathbf{R}^d; \mathbf{R}^d)$.

For a Fréchet space χ , we denote by $D([0, T]; \chi)$ the space of χ -valued càdlàg functions on $[0, T]$ and by $C([0, T]^2; \chi)$ the space of χ -valued continuous functions on $[0, T] \times [0, T]$. The spaces $D([0, T]; \chi)$ and $C([0, T]^2; \chi)$ are endowed with the supremum semi-norms.

The notation $N = N(\cdot, \dots, \cdot)$ is used to denote a positive constant depending only on the quantities appearing in the parentheses. In a given context, the same letter is often used to denote different constants depending on the same parameter. If we do not specify to which space the parameters ω, t, x, y, z and n belong, then we mean $\omega \in \Omega$, $t \in [0, T]$, $x, y \in \mathbf{R}^d$, $z \in Z$, and $n \in \mathbf{N}$.

Let $r_1(x) = \sqrt{1 + |x|^2}$, $x \in \mathbf{R}^d$. For each real number $\beta > 1$, we introduce the following regularity condition on the coefficients b, σ , and H .

Assumption 2.1 (β). (1) *There is a constant $N_0 > 0$ such that for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$,*

$$|r_1^{-1}b_t|_0 + |\nabla b_t|_{\beta-1} + |r_1^{-1}\sigma_t|_0 + |\nabla \sigma_t|_{\beta-1} \leq N_0 \quad \text{and} \quad |r_1^{-1}H_t(z)|_0 + |\nabla H_t(z)|_{\beta-1} \leq K_t(z),$$

where $K : \Omega \times [0, T] \times Z \rightarrow \mathbf{R}_+$ is a $\mathcal{P}_T \otimes \mathcal{Z}$ -measurable function satisfying

$$K_t(z) + \int_Z K_t(z)^2 \pi(dz) \leq N_0,$$

for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$.

(2) *There are constants $\eta \in (0, 1)$ and $N_\kappa > 0$ such that for all $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^d \times Z : |\nabla H_t(\omega, x, z)| > \eta\}$,*

$$|(I_d + \nabla H_t(x, z))^{-1}| \leq N_\kappa.$$

The following theorem shows that if Assumption 2.1 (β) holds for some $\beta > 1$, then for any $\beta' \in [1, \beta]$, the solution $X_t(\tau, x)$ of (1.1) has a modification that is a $C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d)$ -diffeomorphism and the p -th moments of the weighted β' -Hölder norms of the inverse flow are bounded. This theorem will be proved in the next section.

Theorem 2.1. *Let Assumption 2.1(β) hold for some $\beta > 1$.*

(1) *For each stopping time $\tau \leq T$ and $\beta' \in [1, \beta)$, there exists a modification of the strong solution $X_t(\tau, x)$ of (1.1), also denoted by $X_t(\tau, x)$, such that \mathbf{P} -a.s. the mapping $X_t(\tau, \cdot) : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a $C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d)$ -diffeomorphism, $X_t(\tau, \cdot), X_t^{-1}(\tau, \cdot) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$, and $X_t^{-1}(\tau, \cdot)$ coincides with the inverse of $X_{t-}(\tau, \cdot)$. Moreover, for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that*

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} X_t(\tau)|_0^p \right] + \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t(\tau)|_{\beta'-1}^p \right] \leq N$$

and a constant $N = N(d, p, N_0, T, \beta', \eta, N_\kappa, \epsilon)$ such that

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} X_t^{-1}(\tau)|_0^p \right] + \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{-1}(\tau)|_{\beta'-1}^p \right] \leq N. \quad (2.1)$$

(2) If $H \equiv 0$, then for each $\beta' \in (1, \beta)$, \mathbf{P} -a.s. $X(\cdot, \cdot), X^{-1}(\cdot, \cdot) \in C([0, T]^2; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$ and for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that

$$\mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-(1+\epsilon)} X_t(s)|_0^p \right] + \mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-\epsilon} \nabla X_t(s)|_{\beta'-1}^p \right] \leq N$$

and

$$\mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-(1+\epsilon)} X_t^{-1}(s)|_0^p \right] + \mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-\epsilon} \nabla X_t^{-1}(s)|_{\beta'-1}^p \right] \leq N.$$

Remark 2.2. The estimate (2.1) is used in [LM14] to take the optional projection of a linear transformation of the inverse flow of a jump SDE driven by two independent Wiener processes and two independent Poisson random measures relative to the filtration generated by one of the Wiener processes and Poisson random measures.

Now, let us state our strong limit theorem for a sequence of flows, which will also be proved in the next section. We will use this strong limit theorem in [LM14] to show that the inverse flow of a jump SDE solves a parabolic stochastic integro-differential equation. For each n , consider the stochastic flow $X_t^{(n)} = X_t^{(n)}(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$, generated by the SDE

$$dX_t^{(n)} = b_t^{(n)}(X_t^{(n)})dt + \sigma_t^{(n)l_0}(X_t^{(n)})dw_t^0 + \int_Z H_t^{(n)}(X_t^{(n)}, z)q(dt, dz), \quad \tau \leq t \leq T,$$

$$X_t^{(n)} = x, \quad t \leq \tau.$$

Here we assume that for each n , $b^{(n)}$, $\sigma^{(n)}$, and $H^{(n)}$ satisfy the same measurability conditions as b , σ , and H , respectively.

Theorem 2.3. *Let Assumption 2.1(β) hold for some $\beta > 1$ and assume that $b^{(n)}$, $\sigma^{(n)}$, and $H^{(n)}$ satisfy Assumption 2.1 (β) uniformly in $n \in \mathbf{N}$. Moreover, assume that*

$$d\mathbf{P}dt - \lim_{n \rightarrow \infty} \left(|r_1^{-1} b_t^{(n)} - r_1^{-1} b_t|_0 + |\nabla b_t^{(n)} - \nabla b_t|_{\beta-1} \right) = 0,$$

$$d\mathbf{P}dt - \lim_{n \rightarrow \infty} \left(|r_1^{-1} \sigma_t^{(n)} - r_1^{-1} \sigma_t|_{\beta-1} + |\nabla \sigma_t^{(n)} - \nabla \sigma_t|_0 \right) = 0,$$

and for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$ and $n \in \mathbf{N}$,

$$|r_1^{-1} H_t^{(n)}(z) - r_1^{-1} H_t(z)|_0 + |\nabla H_t^{(n)}(z) - \nabla H_t(z)|_{\beta-1} \leq K^{(n)}(t, z),$$

where $(K_t^{(n)}(z))_{n \in \mathbf{N}}$ is a sequence of \mathbf{R}_+ -valued $\mathcal{P}_T \otimes \mathcal{Z}$ measurable functions defined on $\Omega \times [0, T] \times Z$ satisfying for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$ and $n \in \mathbf{N}$,

$$K_t^{(n)}(z) + \int_Z K_t^{(n)}(z)^2 \pi(dz) \leq N_0$$

and

$$d\mathbf{P}dt - \lim_{n \rightarrow \infty} \int_Z K_t^{(n)}(z)^2 \pi(dz) = 0.$$

Then for each stopping time $\tau \leq T$, $\beta' \in [1, \beta)$, $\epsilon > 0$, and $p \geq 2$, we have

$$\lim_{n \rightarrow \infty} \left(\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} X_t^{(n)}(\tau) - r_1^{-(1+\epsilon)} X_t(\tau)|_0^p \right] + \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{(n)}(\tau) - r_1^{-\epsilon} \nabla X_t(\tau)|_{\beta'-1}^p \right] \right) = 0,$$

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} X_t^{(n);-1}(\tau) - r_1^{-(1+\epsilon)} X_t^{-1}(\tau)|_0^p \right] = 0,$$

and

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{(n);-1}(\tau) - r_1^{-\epsilon} \nabla X_t^{-1}(\tau)|_{\beta'-1}^p \right] = 0.$$

Let us introduce our class of solutions for the equation (1.1). For a each number $\beta' > 2$, let $\mathfrak{C}_{cts}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d)$ be the linear space of all random fields $v : \Omega \times [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}^d$ such that v is $\mathcal{O}_T \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable and \mathbf{P} -a.s. $r_1^{-\lambda}(\cdot)v(\cdot)$ is a $C([0, T]; \mathcal{C}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$ for a real number $\lambda > 0$.

We introduce the following assumption for a real number $\beta > 2$.

Assumption 2.2 (β). *There is a constant N_0 such that for all $(\omega, t) \in \Omega \times [0, T]$,*

$$|r_1^{-1}b_t|_0 + |r_1^{-1}\sigma_t|_0 + |\nabla b_t|_{\beta-1} + |\nabla \sigma_t|_{\beta-1} \leq N_0.$$

Theorem 2.4. *Let Assumption 2.2(β) hold for some $\beta > 2$. Then for each stopping time $\tau \leq T$ and $\beta' \in [1, \beta)$, there exists a unique process $u(\tau)$ in $\mathfrak{C}_{cts}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d)$ that solves (1.2). Moreover, \mathbf{P} -a.s. $u_t(\tau, x) = Y_t^{-1}(\tau, x)$ for all $(t, x) \in [0, T] \times \mathbf{R}^d$ and for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that*

$$\mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-(1+\epsilon)} u_t(s)|_0^p \right] + \mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-\epsilon} \nabla u_t(s)|_{\beta'-1}^p \right] \leq N.$$

Remark 2.5. It is clear by the proof of this theorem that if $\sigma \equiv 0$, then we only need to assume that Assumption 2.2 (β) holds for some $\beta > 1$.

Now, consider the SPDE given by

$$d\bar{u}_t(x) = \left(\frac{1}{2} \sigma_t^{i\bar{o}}(x) \sigma_t^{j\bar{o}}(x) \partial_{ij} \bar{u}_t(x) + b_t^i(x) \partial_i \bar{u}_t(x) \right) dt + \sigma_t^{i\bar{o}}(x) \partial_i \bar{u}_t(x) dw_t^{\bar{o}}, \quad \tau < t \leq T,$$

$$\bar{u}_t(x) = x, \quad t \leq \tau. \tag{2.2}$$

This SPDE differs from the one given in (1.2) by the first-order coefficient in the drift. In order to obtain an existence and uniqueness theorem for this equation, we have to impose additional assumptions on σ .

We introduce the following assumption for a real number $\beta > 2$.

Assumption 2.3 (β). *There is a constant $N_0 > 0$ such that for all $(\omega, t) \in \Omega \times [0, T]$,*

$$|r_1^{-1}b_t|_0 + |\nabla b_t|_{\beta-1} + |\sigma_t|_{\beta+1} \leq N_0.$$

For each $\tau \leq T$, consider the stochastic flow $\hat{Y}_t = \hat{Y}_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$, generated by the SDE

$$\begin{aligned} d\bar{Y}_t &= -\hat{b}_t(\bar{Y}_t)dt - \sigma_t^\theta(\bar{Y}_t)dw_t^\theta, \quad \tau < t \leq T, \\ Y_t &= x, \quad t \leq \tau. \end{aligned}$$

If Assumption 2.3(β) holds for some $\beta > 2$, then for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbf{R}^d$,

$$|\hat{b}_t(x)| \leq |b_t(x)| + |\sigma_t(x)|\|\nabla \sigma_t(x)\| \leq N_0(N_0 + 1) + N_0|x|$$

and

$$|\nabla \hat{b}_t|_{\beta-1} \leq |\nabla b_t|_{\beta-1} + |\sigma_t|_{\beta-1}|\nabla^2 \sigma_t|_{\beta-1} + |\nabla \sigma_t|_{\beta-1}^2 \leq N_0 + 2N_0^2,$$

which immediately implies the following corollary of Theorem 2.4.

Corollary 2.6. *Let Assumption 2.3(β) hold for some $\beta > 2$. Then for each stopping time $\tau \leq T$ and $\beta' \in [1, \beta)$, there exists a unique process $\bar{u}(\tau)$ in $\mathfrak{C}_{cts}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d)$ that solves (2.2). Moreover, \mathbf{P} -a.s. $\bar{u}_t(\tau, x) = \bar{Y}_t^{-1}(\tau, x)$ for all $(t, x) \in [0, T] \times \mathbf{R}^d$ and for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that*

$$\mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-(1+\epsilon)} \bar{u}_t(s)|_0^p \right] + \mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-\epsilon} \nabla \bar{u}_t(s)|_{\beta'-1}^p \right] \leq N.$$

3 Properties of stochastic flows

3.1 Homeomorphism property of flows

In this subsection, we collect some results about flows of jump SDEs that we will need. In particular, we present sufficient conditions that guarantee the homeomorphism property of flows of jump SDEs. First, let us introduce the following assumption, which is the usual linear growth and Lipschitz condition on the coefficients b, σ , and H of the SDE (1.1).

Assumption 3.1. *There is a constant $N_0 > 0$ such that for all $(\omega, t, x, y) \in \Omega \times [0, T] \times \mathbf{R}^{2d}$,*

$$\begin{aligned} |b_t(x)| + |\sigma_t(x)| &\leq N_0(1 + |x|), \\ |b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| &\leq N_0|x - y|. \end{aligned}$$

Moreover, for all $(\omega, t, x, y, z) \in \Omega \times [0, T] \times \mathbf{R}^{2d} \times Z$,

$$\begin{aligned} |H_t(x, z)| &\leq K_1(t, z)(1 + |x|), \\ |H_t(x, z) - H_t(y, z)| &\leq K_2(t, z)|x - y|, \end{aligned}$$

where $K_1, K_2 : \Omega \times [0, T] \times Z \rightarrow \mathbf{R}_+$ are $\mathcal{P}_T \otimes \mathcal{Z}$ -measurable functions satisfying

$$K_1(t, z) + K_2(t, z) + \int_Z (K_1(t, z)^2 + K_2(t, z)^2) \pi(dz) \leq N_0,$$

for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$.

It is well-known that under this assumption that there exists a unique strong solution $X_t(s, x)$ of (1.1) (see e.g. Theorem 3.1 in [Kun04]). We will also make use of the following assumption.

Assumption 3.2. *For all $(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^d \times Z$, $H_t(x, z)$ is differentiable in x , and there are constants $\eta \in (0, 1)$ and $N_\kappa > 0$ such that for all $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^d \times Z : |\nabla H_t(\omega, x, z)| > \eta\}$,*

$$|(I_d + \nabla H_t(x, z))^{-1}| \leq N_\kappa.$$

The coming lemma shows that under Assumptions 3.1 and 3.2, the mapping $x + H_t(x, z)$ from \mathbf{R}^d to \mathbf{R}^d is a diffeomorphism and the gradient of inverse map is bounded.

Lemma 3.1. *Let Assumptions 3.1 and 3.2 hold. For each $(\omega, t, z) \in \Omega \times [0, T] \times Z$, the mapping $\tilde{H}_t(\cdot, z) : \mathbf{R}^d \rightarrow \mathbf{R}^d$ defined by $\tilde{H}_t(x, z) := x + H_t(x, z)$ is a diffeomorphism and*

$$|\tilde{H}_t^{-1}(x, z)| \leq \bar{N}N_0 + \bar{N}|x| \quad \text{and} \quad |\nabla \tilde{H}_t^{-1}(x, z)| \leq \bar{N},$$

where $\bar{N} := (1 - \eta)^{-1} \vee N_0$.

Proof. (1) On the set $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^d \times Z : |\nabla H_t(\omega, x, z)| \leq \eta\}$, we have

$$|\kappa_t(\omega, x, z)| \leq \left| I_d + \sum_{n=1}^{\infty} (-1)^n [\nabla H_t(\omega, x, z)]^n \right| \leq \frac{1}{1 - \eta}.$$

It follows from Assumption 3.2 that for all ω, t, x , and z , the mapping $\nabla \tilde{H}_t(x, z)$ has a bounded inverse. Therefore, by Theorem 0.2 in [DHI13] the mapping $\tilde{H}_t(\cdot, z) : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a global diffeomorphism. Moreover, for all ω, t, x and z ,

$$|\tilde{H}_t^{-1}(x, z) - \tilde{H}_t^{-1}(y, z)| \leq \bar{N}|x - y|,$$

which yields

$$|\tilde{H}_t(x, z) - \tilde{H}_t(y, z)| \geq \bar{N}^{-1}|x - y| \implies |\tilde{H}_t(x, z)| + K_1(t, z) \geq \bar{N}^{-1}|x|,$$

and hence

$$|\tilde{H}_t^{-1}(x, z)| \leq \bar{N}K_1(t, z) + \bar{N}|x| \leq \bar{N}N_0 + \bar{N}|x|.$$

□

The following estimates are essential in the proof of the homeomorphic property of the flow and the derivation of moment estimates of the inverse flow. We refer the reader to Theorem 3.2 and Lemmas 3.7 and 3.9 in [Kun04] and Lemma 4.5.6 in [Kun97] ($H \equiv 0$ case) for the proof of the following lemma.

Lemma 3.2. *Let Assumption 3.1 hold.*

- (1) For each $p \geq 2$, there is a constant $N = N(p, N_0, T)$ such that for all $s, \bar{s} \in [0, T]$ and $x, y \in \mathbf{R}^d$,

$$\mathbf{E} \left[\sup_{t \leq T} r_1(X_t(s, x))^p \right] \leq N r_1(x)^p, \quad (3.1)$$

$$\mathbf{E} \left[\sup_{t \leq T} |X_t(s, x) - X_t(s, y)|^p \right] \leq N |x - y|^p. \quad (3.2)$$

- (2) If Assumption 3.2 holds, then for each $p \in \mathbf{R}$, there is a constant $N = N(p, N_0, T, \eta, N_\kappa)$ such that for all $s \in [0, T]$ and $x, y \in \mathbf{R}^d$,

$$\mathbf{E} \left[\sup_{t \leq T} r_1(X_t(s, x))^p \right] \leq N r_1(x)^p, \quad (3.3)$$

and

$$\mathbf{E} \left[\sup_{t \leq T} |X_t(s, x) - X_t(s, y)|^p \right] \leq N |x - y|^p. \quad (3.4)$$

In the next proposition, we collect some facts about the homeomorphic property of the flow. Actually, the homeomorphism property has been shown in [QZ08] to hold under the log-Lipschitz condition (i.e. one uses Bihari's inequality instead of Gronwall's inequality), but we do not pursue this here.

Proposition 3.3. *Let Assumptions 3.1 and 3.2 hold.*

- (1) There exists a modification of the strong solution $X_t(s, x)$, $(s, t, x) \in [0, T]^2 \times \mathbf{R}^d$, of (1.1), also denoted by $X_t(s, x)$, that is càdlàg in s and t and continuous in x . Moreover, for each stopping time $\tau \leq T$, \mathbf{P} -a.s. for all $t \in [0, T]$, the mappings $X_t(\tau, \cdot), X_{t-}(\tau, \cdot) : \mathbf{R}^d \rightarrow \mathbf{R}^d$ are homeomorphisms and the inverse of $X_t(\tau, \cdot)$, denoted by $X_t^{-1}(\tau, \cdot)$, is càdlàg in t and continuous in x , and $X_{t-}^{-1}(\tau, \cdot)$ coincides with the inverse of $X_{t-}(\tau, \cdot)$. In particular, if $(x_n)_{n \geq 1}$ is a sequence in \mathbf{R}^d such that $\lim_{n \rightarrow \infty} x_n = x$ for some $x \in \mathbf{R}^d$, then \mathbf{P} -a.s.

$$\lim_{n \rightarrow \infty} \sup_{t \leq T} |X_t^{-1}(\tau, x_n) - X_t^{-1}(\tau, x)| = 0.$$

Furthermore, for each $\beta' \in [0, 1]$, \mathbf{P} -a.s. $X(\tau, \cdot) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$ and for all $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} X_t(\tau)|_{\beta'}^p \right] \leq N. \quad (3.5)$$

- (2) If $H \equiv 0$, then \mathbf{P} -a.s. for all $s, t \in [0, T]$, the $X_t(s, x)$ and $X_t^{-1}(s, x)$ are continuous in s, t , and x . Moreover, for each $\beta' \in [0, 1]$, \mathbf{P} -a.s. $X(\cdot, \cdot) \in C([0, T]^2; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$ and for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that

$$\mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-(1+\epsilon)} X_t(s)|_{\beta'}^p \right] \leq N. \quad (3.6)$$

Proof. (1) Owing to Assumptions 3.1 and 3.2, by Lemma 3.1, for all ω, t and z , the process $\tilde{H}_t(x, z) := x + H_t(x, z)$ is a homeomorphism (in fact, it is a diffeomorphism) in x and $\tilde{H}_t^{-1}(x, z)$ has linear growth and is Lipschitz. This implies that assumptions of Theorem 3.5 in [Kun04] hold and hence there is modification of $X_t(s, x)$, denoted $\bar{X}_t(s, x)$, such that for all $s \in [0, T]$, \mathbf{P} -a.s. for all $t \in [0, T]$, $X_t(s, \cdot)$ is a homeomorphism. Following [Kun04], for each $(s, t, x) \in [0, T]^2 \times \mathbf{R}^d$, we set

$$\bar{X}_t(s, x) = \begin{cases} x & t \leq s \\ X_t(0, X_s^{-1}(0, x)) & t \geq s, \end{cases} \quad (3.7)$$

and remark that \mathbf{P} -a.s. $\bar{X}_t(s, x)$ is càdlàg in s and t and continuous in x , and \mathbf{P} -a.s. for all $(s, t) \in [0, T]^2$, $\bar{X}_t(s, \cdot)$ is a homeomorphism, and $\bar{X}_t(s, x)$ is a version of $X_t(s, x)$ (the equation started at s). Fix a stopping time $\tau \leq T$. We will now show that $\bar{X}_t(\tau, x) = \bar{X}_t(s, x)|_{s=\tau}$ (i.e. $\bar{X}_t(s, x)$ evaluated at $s = \tau$) is a version of $X_t(\tau, x)$. Define the sequence of stopping times $(\tau_n)_{n \geq 1}$ by

$$\tau_n = \sum_{k=1}^{n-1} \frac{kT}{n} \mathbf{1}_{\{(k-1)T/n \leq \tau < kT/n\}} + T \mathbf{1}_{\{\tau \geq (n-1)T/n\}}.$$

For each n and x , let $X_t^{(n)} = X_t^{(n)}(x) = \bar{X}_t(\tau_n, x)$, $t \in [0, T]$. It follows that for each n, t , and x , \mathbf{P} -a.s. for all $k \in \{1, \dots, n\}$,

$$X_t^{(n)}(x) \mathbf{1}_{\{\tau_n = kT/n\}} = X_t\left(\frac{kT}{n}, x\right) \mathbf{1}_{\{\tau_n = kT/n\}},$$

and hence

$$\begin{aligned} X_t^{(n)}(x) \mathbf{1}_{\{\tau_n = kT/n\}} &= \mathbf{1}_{\{\tau_n = kT/n\}} x + \mathbf{1}_{\{\tau_n = kT/n\}} \int_{]kT/n, kT/n \vee t]} b_r(X_r^{(n)}(x)) dr \\ &\quad + \mathbf{1}_{\{\tau_n = kT/n\}} \int_{]kT/n, kT/n \vee t]} \sigma_r^Q(X_r^{(n)}(x)) dW_r^Q \\ &\quad + \mathbf{1}_{\{\tau_n = kT/n\}} \int_{]kT/n, kT/n \vee t]} \int_Z H_r(X_r^{(n)}(x), z) q(dr, dz). \end{aligned}$$

Since Ω is the disjoint union of the sets $\{\tau_n = kT/n\}$, $k \in \{1, \dots, n\}$, it follows that $X_t^{(n)}(x)$ solves

$$\begin{aligned} X_t^{(n)}(x) &= x + \int_{] \tau_n, \tau_n \vee t]} b_r(X_r^{(n)}(x)) dr + \int_{] \tau_n, \tau_n \vee t]} \sigma_r^Q(X_r^{(n)}(x)) dW_r^Q \\ &\quad + \int_{] \tau_n, \tau_n \vee t]} \int_Z H_r(X_r^{(n)}(x), z) q(dr, dz). \end{aligned}$$

Thus, by uniqueness, we have that for each t and x , \mathbf{P} -a.s. $\bar{X}_t(\tau_n, x) = X_t^{(n)}(x) = X_t(\tau_n, x)$. It is easy to check that for each t and x , \mathbf{P} -a.s. $X_t(\tau_n, x)$ converges to $X_t(\tau, x)$ as n tends to infinity. Since $\bar{X}_t(s, x)$ is càdlàg in s , we have that $\bar{X}_t(\tau_n, x)$ converges to $\bar{X}_t(\tau, x)$ as n tends to infinity. Therefore, $\bar{X}_t(\tau, x)$ is a version of $X_t(\tau, x)$ for all t and x . We identify $X_t(s, x)$ and $\bar{X}_t(s, x)$

for all $(s, t, x) \in [0, T]^2 \times \mathbf{R}^d$. Using Lemma 3.2(1) and Corollary 5.3, we obtain that \mathbf{P} -a.s. $X(\tau, \cdot) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$ and that the estimate (3.5) holds. Note here that for each $\beta \geq 0$, the Fréchet spaces $D([0, T]; C_{loc}^{\beta}(\mathbf{R}^d; \mathbf{R}^d))$ and $C_{loc}^{\beta}(\mathbf{R}^d; D([0, T]; \mathbf{R}^d))$ are equivalent. It follows from the proof of Theorem 3.5 in [Kun04] that for every stopping time $\bar{\tau} \leq T$, \mathbf{P} -a.s.

$$\lim_{|x| \rightarrow \infty} \inf_{t \leq T} |X_t(\bar{\tau}, x)| = \infty. \quad (3.8)$$

Let $(t_n) \subseteq [0, T]$ and $(x_n) \subseteq \mathbf{R}^d$ be convergent sequences with limits t and x , respectively. First, assume $t_n < t$ for all n . By (3.8), for every stopping time $\bar{\tau} \leq T$, \mathbf{P} -a.s. the sequence $(X_{t_n}^{-1}(\bar{\tau}, x_n))$ is uniformly bounded. Since \mathbf{P} -a.s. $X(\tau, \cdot) \in D([0, T]; C^{\beta}(\mathbf{R}^d; \mathbf{R}^d))$, $\beta' \in (0, 1)$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} (X_{t_n}(\bar{\tau}, X_{t_n}^{-1}(\bar{\tau}, x_n)) - X_{t_n}(\bar{\tau}, X_{t_n}^{-1}(\bar{\tau}, x))) &= \lim_{n \rightarrow \infty} (X_{t_n}(\bar{\tau}, X_{t_n}^{-1}(\bar{\tau}, x_n)) - x) \\ &= \lim_{n \rightarrow \infty} (X_{t_n}(\bar{\tau}, X_{t_n}^{-1}(\bar{\tau}, x_n)) - x) = \lim_{n \rightarrow \infty} (x_n - x) = 0, \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} X_{t_n}^{-1}(\bar{\tau}, x_n) = X_{t_n}^{-1}(\bar{\tau}, x).$$

A similar argument is used for $t_n > t$. (2) It follows from the definition (3.7) that $\bar{X}_t(s, x)$ and $\bar{X}_t^{-1}(s, x)$ are continuous in s, t , and x . Moreover, applying Lemma 3.2(1) and Corollary 5.3, we get that \mathbf{P} -a.s. $X(\cdot, \cdot) \in C([0, T]^2; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$ and that the estimate (3.6) holds. The continuity of $X_s(\tau, x)$ with respect to s actually plays an important role in the proof of Theorem 2.4. \square

3.2 Moment estimates of inverse flows: Proof of Theorem 2.1

In this subsection, under Assumption 2.1 (β), $\beta \geq 1$, we derive moment estimates for the flow $X_t(\tau, x)$ and its inverse $X_t^{-1}(\tau, x)$ in weighted Hölder norms and complete the proof of Theorem 2.1. In particular, we will apply Corollaries 5.2 and 5.3 with the Banach spaces $V = D([0, T]; \mathbf{R}^d)$ and $V = C([0, T]^2; \mathbf{R}^d)$.

Proposition 3.4. *Let Assumption 2.1(β) hold for some $\beta > 1$*

- (1) *For each stopping time $\tau \leq T$ and $\beta' \in [1, \beta)$, \mathbf{P} -a.s. $\nabla X(\tau, \cdot) \in D([0, T]; C_{loc}^{\beta'-1}(\mathbf{R}^d; \mathbf{R}^d))$ and for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that*

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t(\tau)|_{\beta'-1}^p \right] \leq N. \quad (3.9)$$

Moreover, for each $p \geq 2$, there is a constant $N = N(d, p, N_0, \beta, T)$ such that for all multi-indices γ with $1 \leq |\gamma| \leq [\beta]$ and all $x \in \mathbf{R}^d$,

$$\mathbf{E} \left[\sup_{t \leq T} |\partial^\gamma X_t(\tau, x)|^p \right] \leq N \quad (3.10)$$

and for all multi-indices γ with $|\gamma| = [\beta]^-$ and all $x, y \in \mathbf{R}^d$,

$$\mathbf{E} \left[\sup_{t \leq T} |\partial^\gamma X_t(\tau, x) - \partial^\gamma X_t(\tau, y)|^p \right] \leq N |x - y|^{([\beta]^+)^p}. \quad (3.11)$$

(2) If $H \equiv 0$, then for each $\beta' \in [1, \beta)$, \mathbf{P} -a.s. $\nabla X(\cdot, \cdot) \in C([0, T]^2; C_{loc}^{\beta'-1}(\mathbf{R}^d; \mathbf{R}^d))$ and for each $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that

$$\mathbf{E} \left[\sup_{s, t \leq T} |r_1^{-(1+\epsilon)} \nabla X_t(s)|_{\beta'-1}^p \right] \leq N.$$

Moreover, for each $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta)$ such that for all multi-indices γ with $|\gamma| = [\beta]^-$ and all $s, \bar{s} \in [0, T]$ and $x \in \mathbf{R}^d$,

$$\mathbf{E} \left[\sup_{t \leq T} |\partial^\gamma X_t(s, x) - \partial^\gamma X_t(\bar{s}, x)|^p \right] \leq N |s - \bar{s}|^{p/2}. \quad (3.12)$$

Proof. (1) Fix a stopping time $\tau \leq T$ and write $X_t(\tau, x) = X_t(x)$. First, let us assume that $[\beta]^- = 1$. It follows from Theorem 3.4 in [Kun04] that \mathbf{P} -a.s. for all t , $X_t(\tau, \cdot)$ is continuously differentiable and $U_t = \nabla X_t(\tau, x)$ satisfies

$$\begin{aligned} dU_t &= \nabla b_t(X_t) U_t dt + \nabla \sigma_t^\rho(X_{t-}) U_t dW_t^\rho + \int_Z \nabla H_t(X_{t-}, z) U_{t-} q(dt, dz), \quad \tau < t \leq T, \\ \nabla X_t &= I_d, \quad t \leq \tau, \end{aligned} \quad (3.13)$$

where I_d is the $d \times d$ -dimensional identity matrix. Taking $\lambda = 0$ in the estimates (3.10) and (3.11) in Theorem 3.3 in [Kun04], we obtain (3.10) and (3.11). Then applying Corollary 5.3 with $V = D([0, T]; \mathbf{R}^d)$, we have that $X(\cdot) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$ and that (3.9) holds. The proof for $[\beta]^- > 1$ follows by induction (see, e.g. the proof of Theorem 6.4 in [Kun97]).

(2) The estimate (3.12) is given in Theorem 4.6.4 in [Kun97] in equation (19). The remaining items of part (2) then follow in exactly the same way as part (1) with the only exception being that we apply Corollary 5.3 with $V = C([0, T]^2; \mathbf{R}^d)$. \square

Lemma 3.5. *Let Assumption 2.1(β) hold for some $\beta > 1$.*

(1) *For each stopping time $\tau \leq T$ and $\beta' \in [1, \beta)$, \mathbf{P} -a.s. $\nabla X(\tau, \cdot)^{-1} \in D([0, T]; C_{loc}^{\beta'-1}(\mathbf{R}^d; \mathbf{R}^d))$ and for each $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \eta, N_\kappa)$ such that for all $x, y \in \mathbf{R}^d$*

$$\mathbf{E} \left[\sup_{t \leq T} |\nabla X_t(\tau, x)^{-1}|^p \right] \leq N \quad (3.14)$$

and

$$\mathbf{E} \left[\sup_{t \leq T} |\nabla X_t(\tau, x)^{-1} - \nabla X_t(\tau, y)^{-1}|^p \right] \leq N |x - y|^{((\beta-1) \wedge 1)p}. \quad (3.15)$$

- (2) If $H \equiv 0$, then for each $\beta' \in [1, \beta)$, \mathbf{P} -a.s. $\nabla X(\cdot, \cdot)^{-1} \in C([0, T]^2; C_{loc}^{\beta'-1}(\mathbf{R}^d; \mathbf{R}^d))$ and for each $p \geq 2$, there is a constant $N = N(d, p, N_0, T)$ such that for all $s, \bar{s} \in [0, T]$ and $x \in \mathbf{R}^d$,

$$\mathbf{E} \left[\sup_{t \leq T} |\nabla X_t(s, x)^{-1} - \nabla X_t(\bar{s}, x)^{-1}|^p \right] \leq N |s - \bar{s}|^{p/2}.$$

Proof. (1) Let $\tau \leq T$ be a fixed stopping time and write $X_t(\tau, x) = X_t(x)$. Using Itô's formula (see also Lemma 3.12 in [Kun04]), we deduce that $\bar{U}_t = [\nabla X_t(x)]^{-1}$ satisfies

$$\begin{aligned} d\bar{U}_t &= \bar{U}_t (\nabla \sigma_t^0(X_{t-}) \nabla \sigma_t^0(X_{t-}(\tau)) - \nabla b_t(X_t)) dt - \bar{U}_t \nabla \sigma_t^0(X_t) dw_t^0 \\ &\quad - \int_Z \bar{U}_{t-} \nabla H_t(X_{t-}, z) (I_d + \nabla H_t(X_{t-}, z))^{-1} q(dt, dz) \\ &\quad + \int_Z \bar{U}_t \nabla H_t(X_{t-}, z)^2 (I_d + \nabla H_t(X_{t-}, z))^{-1} \pi(dz) dt, \quad \tau < t \leq T, \\ \bar{U}_t &= I_d, \quad t \leq \tau. \end{aligned} \tag{3.16}$$

Since matrix inversion is a smooth mapping, the coefficients of the linear equation (3.16) satisfy the same assumptions as the coefficients of the linear equation (3.13), and hence the derivation of the estimates (3.14) and (3.15) proceed in the same way as the analogous estimates for (3.13). To see that \mathbf{P} -a.s. $X(\cdot)^{-1} \in D([0, T]; C_{loc}^{\beta'-1}(\mathbf{R}^d; \mathbf{R}^d))$, we only need to note that \mathbf{P} -a.s. $X(\cdot) \in D([0, T]; C_{loc}^{\beta'-1}(\mathbf{R}^d; \mathbf{R}^d))$ and that matrix inversion is a smooth mapping. Part (2) follows with the obvious changes. \square

As an immediate corollary, we obtain the diffeomorphism property of the flow $X_t(\tau, x)$ under the assumptions Assumption 2.1(β), $\beta > 1$.

Corollary 3.6. *Let Assumption 2.1(β) hold.*

- (1) For each stopping time $\tau \leq T$ and $\beta' \in [1, \beta)$ the mapping $X_t(\tau, \cdot) : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a $C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d)$ -diffeomorphism, \mathbf{P} -a.s. $X(\cdot, \cdot), X^{-1}(\cdot, \cdot) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$ and for each $t \in [0, T]$, $X_t^{-1}(\tau)$ coincides with the inverse of $X_{t-}(\tau)$.
- (2) If $H \equiv 0$, then for each $\beta' \in [1, \beta)$, \mathbf{P} -a.s. $X(\cdot, \cdot), X^{-1}(\cdot, \cdot) \in C([0, T]^2, C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$.

Proof. (1) Fix a stopping time $\tau \leq T$ and write $X_t(\tau, x) = X_t(x)$. It follows from Propositions 3.3 and 3.4 that \mathbf{P} -a.s. for all t , the mappings $X_t(\cdot), X_{t-}(\cdot) : \mathbf{R}^d \rightarrow \mathbf{R}^d$ are homeomorphisms and $X(\cdot) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$. Moreover, by Lemma 3.5, \mathbf{P} -a.s. for all t and x , the matrix $\nabla X_t(\tau, x)$ has an inverse. Therefore, by Hadamard's Theorem (see, e.g., Theorem 0.2 in [DHI13]), \mathbf{P} -a.s. for all t , $X_t(\cdot)$ is a diffeomorphism. Using the chain rule, \mathbf{P} -a.s. for all t and x ,

$$\nabla X_t^{-1}(x) = \nabla X_t(X_t^{-1}(x))^{-1}. \tag{3.17}$$

Since, by Lemma 3.5, \mathbf{P} -a.s. $[\nabla X(\cdot)]^{-1} \in D([0, T]; C_{loc}^{\beta'-1}(\mathbf{R}^d; \mathbf{R}^d))$ and we know that \mathbf{P} -a.s. for all t , $X_t^{-1}(\cdot)$ is differentiable, it follows from (3.17) that \mathbf{P} -a.s.

$$\nabla X(X_t^{-1}(\cdot))^{-1} \in D([0, T]; C_{loc}^{(\beta'-1) \wedge 1}(\mathbf{R}^d; \mathbf{R}^d)).$$

One then proceeds inductively to complete the proof. Making the obvious changes in the proof of part (1), we obtain part (2). \square

We conclude with a derivation of Hölder moment estimates of the inverse flow $X_t^{-1}(\tau, x)$, which will complete the proof of Theorem 2.1.

Proof of Theorem 2.1. (1) Fix a stopping time $\tau \leq T$ and write $X_t(\tau, x) = X_t(x)$. Fix $\epsilon > 0$. First, let us assume that $[\beta]^- = 1$. Set $J_t(x) = |\det \nabla X_t(x)|$. It is clear from (3.10) that for each $p \geq 2$ and x , there is a constant $N = N(d, p, N_0, T)$ such that

$$\mathbf{E}[\sup_{t \leq T} |J_t(x)|^p] \leq N. \quad (3.18)$$

Using the change of variable $(\bar{x}, \bar{y}) = (X_t^{-1}(x), X_t^{-1}(y))$, Fatou's lemma, Fubini's theorem, Hölder's inequality, and the inequalities (3.3), (3.18), (5), (3.2), and (3.4), for any $\delta \in (0, 1]$ and $p > \frac{d}{\epsilon}$, we obtain that there is a constant $N = N(d, p, N_0, T, \delta, \eta, N_\kappa, \epsilon)$ such that

$$\begin{aligned} \mathbf{E} \sup_{t \leq T} \int_{\mathbf{R}^d} |r_1(x)^{-(1+\epsilon)} X_t^{-1}(x)|^p dx &\leq \int_{\mathbf{R}^d} |\bar{x}|^p \mathbf{E} \sup_{t \leq T} [r_1(X_t(\bar{x}))^{-p(1+\epsilon)} J_t(\bar{x})] d\bar{x} \\ &\leq N \mathbf{E} \int_{\mathbf{R}^d} r_1(\bar{x})^{-p\epsilon} d\bar{x} \leq N \end{aligned}$$

and

$$\begin{aligned} &\mathbf{E} \sup_{t \leq T} \int_{|x-y|<1} \frac{|r_1^{-(1+\epsilon)}(x) X_t^{-1}(x) - r_1^{-(1+\epsilon)}(y) X_t^{-1}(y)|^p}{|x-y|^{2d+\delta p}} dx dy \\ &\leq \int_{|\bar{x}-\bar{y}|<1} \mathbf{E} \sup_{t \leq T} \left[\frac{|r_1^{-p(1+\epsilon)}(X_t(\bar{x}))| \bar{x} - \bar{y}|^p J_t(\bar{x}) J_t(\bar{y})}{|X_t(\bar{x}) - X_t(\bar{y})|^{2d+\delta p}} \right] d\bar{x} d\bar{y} \\ &\quad + \int_{|\bar{x}-\bar{y}|<1} \mathbf{E} \sup_{t \leq T} \left[\frac{|\bar{y}|^p |r_1^{-(1+\epsilon)}(X_t(\bar{x})) - r_1^{-(1+\epsilon)}(X_t(\bar{y}))|^p J_t(\bar{x}) J_t(\bar{y})}{|X_t(\bar{x}) - X_t(\bar{y})|^{2d+\delta p}} \right] d\bar{x} d\bar{y} \\ &\leq N \int_{|\bar{x}-\bar{y}|<1} \frac{r_1(\bar{x})^{-p(1+\epsilon)}}{|\bar{x} - \bar{y}|^{2d-(1-\delta)p}} d\bar{x} d\bar{y} + N \int_{|\bar{x}-\bar{y}|<1} \frac{r_1(\bar{x})^{-p(1+\epsilon)} + r_1(\bar{y})^{-p(1+\epsilon)}}{|\bar{x} - \bar{y}|^{2d-(1-\delta)p}} d\bar{x} d\bar{y} \leq N. \end{aligned}$$

Similarly, making use of the inequalities (3.3), (3.18), (5), (3.2), (3.4), (3.14), and (3.15), for any $p > \frac{d}{\epsilon} \vee \frac{d}{\beta-\beta'} \vee \frac{d}{2-\beta'}$, we get

$$\begin{aligned} \mathbf{E} \sup_{t \leq T} \int_{\mathbf{R}^d} |r^{-\epsilon}(x) \nabla X_t^{-1}(x)|^p dx &\leq \int_{\mathbf{R}^d} \mathbf{E} \sup_{t \leq T} [r_1(X_t(\bar{x}))^{-p\epsilon} |\nabla X_t(\bar{x})|^{-1} J_t(\bar{x})] d\bar{x} \\ &\leq N \mathbf{E} \int_{\mathbf{R}^d} r_1(\bar{x})^{-p\epsilon} d\bar{x} \leq N \end{aligned}$$

and

$$\begin{aligned} &\mathbf{E} \sup_{t \leq T} \int_{|x-y|<1} \frac{|r^{-\epsilon}(x) \nabla X_t^{-1}(x) - r^{-\epsilon}(y) \nabla X_t^{-1}(y)|^p}{|x-y|^{2d+(\beta'-1)p}} dx dy \\ &\leq \int_{|\bar{x}-\bar{y}|<1} \mathbf{E} \sup_{t \leq T} \left[\frac{|r_1^{-\epsilon}(X_t(\bar{x})) [\nabla X_t(\bar{x})]^{-1} - r_1^{-\epsilon}(X_t(\bar{y})) [\nabla X_t(\bar{y})]^{-1}|^p J_t(\bar{x}) J_t(\bar{y})}{|X_t(\bar{x}) - X_t(\bar{y})|^{2d+(\beta'-1)p}} \right] d\bar{x} d\bar{y} \end{aligned}$$

$$\leq N \int_{|\bar{x}-\bar{y}|<1} \frac{r_1(\bar{x})^{-p\epsilon}}{|\bar{x}-\bar{y}|^{2d-(\beta-\beta')p}} d\bar{x}d\bar{y} + N \int_{|\bar{x}-\bar{y}|<1} \frac{r_1(\bar{x})^{-p\epsilon} + r_1(\bar{y})^{-p\epsilon}}{|\bar{x}-\bar{y}|^{2d-(2-\beta')p}} d\bar{x}d\bar{y} \leq N,$$

where $N = N(d, p, N_0, T, \beta', \eta, N_\kappa, \epsilon)$ is a positive constant. Therefore, combining the above estimates and applying Corollary 5.2, we have that for all $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \eta, N_\kappa, \epsilon)$, such that

$$\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} X_t^{-1}(\tau)|_0^p \right] + \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{-1}(\tau)|_{\beta'-1}^p \right] \leq N.$$

It is well-known that the the inverse map \mathfrak{I} on the set of invertible $d \times d$ -dimensional matrices is infinitely differentiable and for each n , there is a constant $N = N(n, d)$ such that for all invertible matrices M , the n th derivative of \mathfrak{I} evaluated at M , denoted $\mathfrak{I}^{(n)}(M)$, satisfies

$$|\mathfrak{I}^{(n)}(M)| \leq N |M|^{-n-1} \leq N |M|^{-1} |M|^{n+1}.$$

We claim that for each n and every multi-index γ with $|\gamma| = n$, the components of $\partial^\gamma X_t^{-1}(x)$ are a polynomial in terms of the entries of $[\nabla X_t(X_t^{-1}(x))]^{-1}$ and $\partial^{\gamma'} \nabla X_t(X_t^{-1}(x))$ for all multi-indices γ' with $1 \leq |\gamma'| \leq n-1$. Assume that statement holds for some n . By the chain rule, for each ω, t , and x , we have

$$\nabla(\nabla X_t(X_t^{-1}(x))^{-1}) = \mathfrak{I}^{(1)}(\nabla X_t(X_t^{-1}(x))) \nabla^2 X_t(X_t^{-1}(x)) \nabla X_t(X_t^{-1}(x))^{-1}$$

and for all multi-indices γ with $1 \leq |\gamma'| \leq n-1$, we have

$$\nabla(\partial^{\gamma'} \nabla X_t(X_t^{-1}(x))) = \partial^{\gamma'} \nabla^2 X_t(X_t^{-1}(x)) \nabla X_t(X_t^{-1}(x))^{-1},$$

where $\nabla^2 X_t(X_t^{-1}(x))$ is the tensor of second-order derivatives of $X_t(\cdot)$ evaluated at $X_t^{-1}(x)$. This implies that for every multi-index γ with $|\gamma| = n+1$, the components of $\partial^\gamma X_t^{-1}(x)$ are a polynomial in terms of the entries of $\nabla X_t(X_t^{-1}(x))^{-1}$ and $\partial^{\gamma'} \nabla X_t(X_t^{-1}(x))$ for all multi-indices γ' with $1 \leq |\gamma'| \leq n$. By induction, the claim is true. Therefore, for $[\beta]^- \geq 2$, using (3.10) and (3.11), we obtain the moment estimates for the inverse flow in the almost exact same way we did for $[\beta]^- = 1$. Making the obvious changes in the proof of part (1), we obtain part (2). This completes the proof of Theorem 2.1. \square

3.3 Strong limit of a sequence of flows: Proof of Theorem 2.3

Proof of Theorem 2.3. Let $\tau \leq T$ be a fixed stopping time and write $X_t(\tau, x) = X_t(x)$. For each n , let

$$Z_t^{(n)}(x) = X_t^{(n)}(x) - X_t(x), \quad (t, x) \in [0, T] \times \mathbf{R}^d.$$

Throughout the proof we denote by $(\delta_n)_{n \geq 1}$ a deterministic sequence with $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ that may change from line to line. Let $N = N(p, N_0, T)$ be a positive constant, which may change from line to line. By virtue of Theorem 2.1 in [Kun04] and (3.1), for all $p \geq 2$ and t, x and n , we have

$$\mathbf{E} \left[\sup_{s \leq t} |Z_s^{(n)}(x)|^p \right] \leq N \mathbf{E} \int_{[0, t]} |Z_s^{(n)}(x)|^p ds + N \delta_n r_1(x)^p.$$

Since the right-hand-side is finite by (3.1), applying Gronwall's lemma we get that for all x and n ,

$$\mathbf{E}[\sup_{t \leq T} |Z_t^{(n)}(x)|^p] \leq N\delta_n r_1(x)^p. \quad (3.19)$$

Similarly, by (3.10), for all x and n , we have

$$\mathbf{E} \left[\sup_{t \leq T} |\nabla Z_t^{(n)}(x)|^p \right] \leq N\delta_n.$$

Using (3.10), for all x, y , and n , we obtain

$$\mathbf{E} \left[\sup_{t \leq T} |Z_t^{(n)}(x) - Z_t^{(n)}(y)|^p \right] \leq |x - y|^p \mathbf{E} \sup_{t \leq T} \int_0^1 |\nabla Z_t^{(n)}(y + \theta(x - y))|^p d\theta \leq N|x - y|^p.$$

It follows immediately from (3.11) that for all x, y , and n ,

$$\mathbf{E}[\sup_{t \leq T} |\nabla Z_t^{(n)}(x) - \nabla Z_t^{(n)}(y)|^p] \leq N|x - y|^{(\beta-1) \vee 1}.$$

Thus, by Corollary 5.4, we have

$$\lim_{n \rightarrow \infty} \left(\mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} X_t^{(n)} - r_1^{-(1+\epsilon)} X_t|_0^p \right] + \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{(n)} - r_1^{-\epsilon} \nabla X_t|_0^p \right] \right) = 0. \quad (3.20)$$

Owing to a standard interpolation inequality for Hölder spaces (see, e.g. Lemma 6.32 in [GT01]), for each $\delta \in (0, 1)$ and $\bar{\beta} \in (\beta', \beta)$, there is a constant $N(\delta)$ such that

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{(n)} - r_1^{-\epsilon} \nabla X_t|_{\beta'-1}^p \right] &\leq \delta \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{(n)} - \nabla X_t|_{\bar{\beta}-1}^p \right] \\ &\quad + C_\delta \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{(n)} - \nabla X_t|_0^p \right], \end{aligned}$$

and hence since

$$\sup_n \mathbf{E} \left[\sup_{t \leq T} |r_1^\epsilon \nabla X_t^{(n)}|_{\bar{\beta}-1}^p \right] + \mathbf{E} \left[\sup_{t \leq T} |r_1^\epsilon \nabla X_t|_{\bar{\beta}-1}^p \right] < \infty,$$

we have

$$\lim_{n \rightarrow \infty} \mathbf{E}[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{(n)} - r_1^{-\epsilon} \nabla X_t|_{\beta'-1}^p] = 0.$$

By Theorem 2.1, Corollary 5.4, and the interpolation inequality for Hölder spaces used above, in order to show

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\sup_{t \leq T} |r_1^{-(1+\epsilon)} X_t^{(n);-1}(\tau) - r_1^{-(1+\epsilon)} X_t^{-1}(\tau)|_0^p \right] = 0$$

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\sup_{t \leq T} |r_1^{-\epsilon} \nabla X_t^{(n);-1}(\tau) - r_1^{-\epsilon} \nabla X_t^{-1}(\tau)|_{\beta'-1}^p \right] = 0,$$

it suffices to show that for each x ,

$$d\mathbf{P} - \lim_{n \rightarrow \infty} \sup_{t \leq T} |X_t^{(n);-1}(x) - X_t^{-1}(x)| = 0 \quad (3.21)$$

and

$$d\mathbf{P} - \lim_{n \rightarrow \infty} \sup_{t \leq T} |\nabla X_t^{(n);-1}(x) - \nabla X_t^{-1}(x)| = 0. \quad (3.22)$$

For each n , define

$$\Theta_t^{(n)}(x) = r_1(X_t^{(n)}(x))^{-1} - r_1(X_t(x))^{-1}, \quad (t, x) \in [0, T] \times \mathbf{R}^d.$$

For all ω, t, x , and n , we have

$$|\Theta_t^{(n)}(x)| \leq r_1(X_t^{(n)}(x))^{-1} r_1(X_t(x))^{-1} |Z_t^{(n)}(x)|,$$

and hence using Hölder's inequality, (3.4), and (3.19), we obtain that for all $p \geq 2$, x , there is a constant $N = N(p, N_0, T, \eta, N_\kappa)$ such that for all n ,

$$\mathbf{E}[\sup_{t \leq T} |\Theta_t^{(n)}(x)|^p] \leq N r_1(x)^{-p} \delta_n,$$

where $N = N(p, N_0, T, \eta, N_\kappa)$ is a constant. Furthermore, since

$$|\nabla \Theta_t^{(n)}(x)| \leq r_1(X_t^{(n)}(x))^{-2} |\nabla X_t^{(n)}(x)| + r_1(X_t(x))^{-2} |\nabla X_t^{(n)}(x)|,$$

for all ω, t, x , and n , applying (3.4) and (3.10), for all $p \geq 2$, x , and n , we get

$$\mathbf{E} \left[\sup_{t \leq T} |r_1(x) \Theta_t^{(n)}(x) - r_1(y) \Theta_t^{(n)}(y)|^p \right] \leq N |x - y|^p.$$

Then owing to Corollary 5.4, for each $p \geq 2$,

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\sup_{t \leq T} |\Theta_t^{(n)}|_0^p \right] = 0. \quad (3.23)$$

We claim that for each $R > 0$,

$$d\mathbf{P} - \lim_{n \rightarrow \infty} E(n, R) := d\mathbf{P} - \lim_{n \rightarrow \infty} \sup_{t \leq T} |X_t^{(n);-1} - X_t^{-1}|_{0; \{|x| \leq R\}} = 0. \quad (3.24)$$

Fix $R > 0$. It is enough to show that every subsequence of $E(n) = E(n, R)$ has a sub-subsequence converging to 0, \mathbf{P} -a.s.. Owing to (3.20) and (3.23), for a given subsequence $(E(n_k))$, we can always find sub-subsequence (still denoted $(E(n_k))$ to avoid double indices) such that \mathbf{P} -a.s.,

$$\lim_{k \rightarrow \infty} \sup_{t \leq T} |X_t^{(n_k)} - X_t|_{\beta'; \{|x| \leq \bar{R}\}} = 0, \quad \forall \bar{R} > 0, \quad (3.25)$$

and

$$\lim_{k \rightarrow \infty} \sup_{t \leq T} |r_1(X_t^{(n_k)}(x))^{-1} - r_1(X_t(x))^{-1}|_0 = 0.$$

Fix an ω for which both limits are zero. We will prove that

$$\lim_{k \rightarrow \infty} \sup_{t \leq T} |X_t^{(n_k);-1}(\omega) - X_t^{-1}(\omega)|_{0; \{|x| \leq R\}} = 0. \quad (3.26)$$

Suppose, by contradiction, that (3.26) is not true. Then there exists an $\varepsilon > 0$ and a subsequence of (n_k) (still denoted (n_k)) such that $t_{n_k} \rightarrow t-$ (or $t_{n_k} \rightarrow t+$) and $x_{n_k} \rightarrow x$ as $k \rightarrow \infty$ with $|x_{n_k}| \leq R$ such that (dropping ω),

$$|X_{t_{n_k}}^{(n_k);-1}(x_{n_k}) - X_{t_{n_k}}^{-1}(x_{n_k})| \geq \varepsilon. \quad (3.27)$$

Arguing by contradiction and using (3.3), we have

$$\sup_k |X_{t_{n_k}}^{(n_k);-1}(x_{n_k})| < \infty. \quad (3.28)$$

Applying (3.28), (3.25), and the fact that $X(\cdot), X^{-1}(\cdot) \in D([0, T]; C_{loc}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d))$, we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} (X_{t-}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k})) - X_{t-}(X_{t_{n_k}}^{-1}(x_{n_k}))) &= \lim_{k \rightarrow \infty} (X_{t-}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k})) - x_{n_k}) \\ &= \lim_{k \rightarrow \infty} (X_{t-}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k})) - X_{t_{n_k}}^{(n_k)}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))) \\ &= \lim_{k \rightarrow \infty} (X_{t-}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k})) - X_{t_{n_k}}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))) \\ &\quad + \lim_{k \rightarrow \infty} (X_{t_{n_k}}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k})) - X_{t_{n_k}}^{(n_k)}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))) = 0, \end{aligned}$$

which contradicts (3.27), and hence proves (3.26), (3.24), and (3.21). For each n , define

$$\bar{U}_t^{(n)} = \bar{U}^{(n)}(t, x) = \nabla X_t^{(n)}(x)^{-1} \quad \text{and} \quad \bar{U}(t) = \bar{U}(t, x) = \nabla X_t(x)^{-1}, \quad (t, x) \in [0, T] \times \mathbf{R}^d.$$

Using (3.14) and (3.15) and repeating the arguments given above, for each $p \geq 2$, we get

$$\lim_n \mathbf{E}[\sup_{t \leq T} |r_1^{-\epsilon} \bar{U}_t^{(n)} - r_1^{-\epsilon} \bar{U}_t|_{\beta'-1}^p] = 0.$$

Then (3.3) and (3.24) imply that for each $R > 0$,

$$\begin{aligned} d\mathbf{P} - \lim_{n \rightarrow \infty} \sup_{t \leq T} |\nabla X_t^{(n);-1}(x) - \nabla X_t^{-1}(x)|_{0; \{|x| \leq R\}} \\ = d\mathbf{P} - \lim_{n \rightarrow \infty} \sup_{t \leq T} |\nabla X_t^{(n)}(X_t^{(n);-1}(x))^{-1} - \nabla X_t(X_t^{-1}(x))^{-1}|_{0; \{|x| \leq R\}} = 0, \end{aligned}$$

which yields (3.22) and completes the proof. \square

4 Classical solution of an SPDE: Proof of Theorem 2.4

Proof of Theorem 2.4. Fix a stopping time $\tau \leq T$. By virtue of Theorem 2.1, we only need to show that $Y^{-1}(\tau) = Y_t^{-1}(\tau, x)$ solves (1.2) and that this is the unique solution. Suppose we have shown $Y^{-1}(s, x)$, $s \in [0, T]$, solves (1.2) (i.e. where τ is deterministic). It is then straightforward to conclude that $Y^{-1}(\tau')$ solves (1.2) for a finite-valued stopping times τ' . We can then use an approximation argument (see the proof of Proposition 3.3) to show that $Y^{-1}(\tau) = Y_t^{-1}(\tau, x)$ solves (1.2). Thus, it suffices to take τ deterministic. Let $u_t(x) = u_t(s, x) = Y_t^{-1}(s, x)$, $(s, t, x) \in [0, T]^2 \times \mathbf{R}^d$. Fix $(s, t, x) \in [0, T]^2 \times \mathbf{R}^d$ with $s < t$ and write $Y_t(x) = Y_t(s, x)$. We will treat a general stopping time $\tau \leq T$ later. Let $((t_n^M)_{0 \leq n \leq M})_{1 \leq M \leq \infty}$ be a sequence of partitions of the interval $[s, t]$ such that for each $M > 0$, $(t_n^M)_{0 \leq n \leq M}$ has mesh size $(t - s)/M$. Fix M and set $(t_n)_{0 \leq n \leq M} = (t_n^M)_{0 \leq n \leq M}$. Immediately, we obtain

$$u_t(x) - x = \sum_{n=0}^{M-1} (u_{t_{n+1}}(x) - u_{t_n}(x)). \quad (4.1)$$

We will use Taylor's theorem to expand each term in the sum on the right-hand-side of (4.1). By Taylor's theorem, for each n and y , we have

$$\begin{aligned} u_{t_{n+1}}(Y_{t_{n+1}}(y)) - u_{t_n}(Y_{t_{n+1}}(y)) &= y - u_{t_n}(Y_{t_{n+1}}(y)) = u_{t_n}(Y_{t_n}(y)) - u_{t_n}(Y_{t_{n+1}}(y)) \\ &= \nabla u_{t_n}(Y_{t_n}(y))(Y_{t_n}(y) - Y_{t_{n+1}}(y)) - (Y_{t_n}(y) - Y_{t_{n+1}}(y))^* \Theta_n(Y_{t_n}(y))(Y_{t_n}(y) - Y_{t_{n+1}}(y)), \end{aligned} \quad (4.2)$$

where

$$\Theta_n^{ij}(z) = \int_0^1 (1 - \theta) \partial_{ij} u_{t_n} \left(z + \theta(Y_{t_{n+1}}(Y_{t_n}^{-1}(z)) - z) \right) d\theta.$$

Since for each n , $Y_{t_{n+1}}(s, x) = Y_{t_{n+1}}(t_n, Y_{t_n}(s, x))$, we have

$$Y_{t_{n+1}}(Y_{t_n}^{-1}(x)) = Y_{t_{n+1}}(t_n, x)$$

and hence substituting $y = Y_{t_n}^{-1}(x)$ into (4.2), for each n , we get

$$u_{t_{n+1}}(x) - u_{t_n}(x) = A_n + B_n, \quad (4.3)$$

where

$$A_n := \nabla u_{t_n}(x)(x - Y_{t_{n+1}}(t_n, x)) - (x - Y_{t_{n+1}}(t_n, x))^* \Theta_n^{ij}(x)(x - Y_{t_{n+1}}(t_n, x))$$

and

$$B_n := (u_{t_{n+1}}(x) - u_{t_n}(x)) - (u_{t_{n+1}}(Y_{t_{n+1}}(t_n, x)) - u_{t_n}(Y_{t_{n+1}}(t_n, x))).$$

Applying Taylor's theorem once more, for each n , we obtain

$$B_n = C_n + D_n, \quad (4.4)$$

where

$$C_n := (\nabla u_{t_{n+1}}(x) - \nabla u_{t_n}(x))(x - Y_{t_{n+1}}(t_n, x)),$$

$$D_n := -(x - Y_{t_{n+1}}(t_n, x))^* \tilde{\Theta}_n(x)(x - Y_{t_{i+1}}(t_i, x)),$$

and

$$\tilde{\Theta}_n(x)^{ij} := \int_0^1 (1 - \theta) \partial_{ij}(u_{t_{n+1}} - u_{t_n})(x + \theta(Y_{t_{n+1}}(t_n, x) - x)) d\theta.$$

Thus, combining (4.1), (4.3), and (4.4), \mathbf{P} -a.s. we have

$$u_t(x) - x = \sum_{n=0}^{M-1} (A_n + C_n + D_n). \quad (4.5)$$

Now, we will derive the limit of the right-hand-side of (4.5).

Claim 4.1. (1)

$$\begin{aligned} d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} A_n &= - \int_{[s, t]} \left[\frac{1}{2} \sigma_r^{i\varrho}(x) \sigma_r^{j\varrho}(x) \partial_{ij} u_r(x) + b_r^i(x) \partial_i u_r(x) \right] dr \\ &\quad - \int_{[s, t]} \sigma_r^{i\varrho}(x) \partial_i u_r(x) dw_r^{\varrho}; \end{aligned}$$

$$(2) \quad d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} D_n = 0;$$

$$(3) \quad d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} C_n = \int_{[s, t]} \sigma_r^{j\varrho}(x) \partial_j \sigma_r^{i\varrho}(x) \partial_i u_r(x) dr + \int_{[s, t]} \sigma_r^{i\varrho}(x) \sigma_r^{j\varrho}(x) \partial_{ij} u_r(x) dr.$$

Proof of Claim 4.1. (1) For each n , we have

$$\begin{aligned} \nabla u_{t_n}(x) (x - Y_{t_{n+1}}(t_n, x)) &= - \int_{[t_n, t_{n+1}]} b_r^i(x) \partial_i u_{t_n}(x) dr - \int_{[t_n, t_{n+1}]} \sigma_r^{i\varrho}(x) \partial_i u_{t_n}(x) dw_r^{\varrho} \\ &\quad + R_n^{(1)} + R_n^{(2)}, \end{aligned}$$

where

$$R_n^{(1)} := \int_{[t_n, t_{n+1}]} \left(b_r^i(x) - b_r^i(Y_r(t_n, x)) \right) \partial_i u_{t_n}(x) dr$$

and

$$R_n^{(2)} := \int_{[t_n, t_{n+1}]} [\sigma_r^{i\varrho}(x) - \sigma_r^{i\varrho}(Y_r(t_n, x))] \partial_i u_{t_n}(x) dw_r^{\varrho}.$$

Since b and σ are Lipschitz, there is a constant $N = N(N_0, T)$ such that

$$\sum_{n=0}^{M-1} |R_n^{(1)}| \leq N \sup_{s \leq r \leq t} |\nabla u_r(x)| \sup_{|r_1 - r_2| \leq \frac{t}{M}} |x - Y_{r_1}(r_2, x)|$$

and

$$\int_{[s, t]} \left| \sum_{n=0}^{M-1} \mathbf{1}_{[t_n, t_{n+1}]}(r) \left(\sigma_r^i(x) - \sigma_r^i(Y_r(t_n, x)) \right) \partial_i u_{t_n}(x) \right|^2 ds$$

$$\leq N \sup_{s \leq r \leq t} |\nabla u_r(x)|^2 \sup_{|r_1 - r_2| \leq \frac{t}{M}} |x - Y_{r_1}(r_2, x)|^2.$$

Owing to the joint continuity of $Y_t(s, x)$ in s and t and the dominated convergence theorem for stochastic integrals, we obtain

$$d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} (R_n^{(1)} + R_n^{(2)}) = 0. \quad (4.6)$$

In a similar way, this time using the continuity of $\nabla u_t(x)$ in t and the linear growth of b and σ , we get

$$\begin{aligned} d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} & \left(- \int_{]t_n, t_{n+1}] } b_r^i(x) \partial_i u_{t_n}(x) dr - \int_{]t_n, t_{n+1}] } \sigma_r^{i\varrho}(x) \partial_i u_{t_n}(x) dw_r^\varrho \right) \\ &= - \int_{]s, t]} b_r(x) \partial_i u_r(x) dr - \int_{]s, t]} \sigma_r^\varrho(x) \partial_i u_r(x) dw_r^\varrho. \end{aligned}$$

For each n , we have

$$-(x - Y_{t_{n+1}}(t_n, x))^* \Theta_n(x) (x - Y_{t_{n+1}}(t_n, x)) = S_n^{(1)} + S_n^{(2)},$$

where $S_n^{(1)}(t, x)$ has only $drdr$ and $drdw_r^\varrho$ terms and where

$$\begin{aligned} S_n^{(2)} := & -\frac{1}{2} \left(\int_{]t_n, t_{n+1}] } \sigma_r^{i\varrho}(Y_r(t_n, x)) dw_r^\varrho \right) \partial_{ij} u_{t_n}(x) \left(\int_{]t_n, t_{n+1}] } \sigma_r^{j\varrho}(Y_r(t_n, x)) dw_r^\varrho \right) \\ & - \left(\int_{]t_n, t_{n+1}] } \sigma_r^{i\varrho}(Y_r(t_n, x)) dw_r^\varrho \right) \left(\Theta_n^{ij}(x) - \frac{1}{2} \partial_{ij} u_{t_n}(x) \right) \left(\int_{]t_n, t_{n+1}] } \sigma_r^{j\varrho}(Y_r(t_n, x)) dw_r^\varrho \right). \end{aligned}$$

Since

$$\begin{aligned} \left| \Theta_n^{ij}(x) - \frac{1}{2} \partial_{ij} u_{t_n}(x) \right| &= \left| \int_0^1 (1 - \theta) (\partial_{ij} u_{t_n}(x + \theta(Y_{t_{n+1}}(t_n, x) - x)) - \partial_{ij} u_{t_n}(x)) d\theta \right| \\ &\leq N \sup_{|r_1 - r_2| \leq \frac{t}{M}, \theta \in (0, 1)} |\partial_{ij} u_{r_1}(x + \theta(Y_{r_2}(r_1, x) - x)) - \partial_{ij} u_{r_1}(x)|, \end{aligned}$$

proceeding as in the derivation of (4.6) and using the joint continuity of $\partial_{ij} u_t(x)$ in t and x , the continuity of $Y_t(s, x)$ in s and t , and standard properties of the stochastic integral (i.e. Thm. 2 (5) in [LS89] and the stochastic dominated convergence theorem), we obtain

$$d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} S_n^{(2)} = -\frac{1}{2} \int_{]0, t]} \sigma_r^{i\varrho}(x) \sigma_r^{j\varrho}(x) \partial_{ij} u_r(x) dr.$$

Similarly, by appealing to standard properties of the stochastic integral and the properties stated in Theorem 2.1(2), we have $d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} S_n^{(1)} = 0$, which completes the proof

of part (1). The proof of part (2) is similar to the proof of part (1), so we proceed to the proof of part (3). We know that for each n , $Y_{t_{n+1}}(x) = Y_{t_{n+1}}(t_n, Y_{t_n}(x))$. Thus, for each n , we have $u_{t_{n+1}}(x) = u_{t_n}(Y_{t_{n+1}}^{-1}(t_n, x))$, and hence by the chain rule,

$$\nabla u_{t_{n+1}}(x) = \nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n, x)) \nabla Y_{t_{n+1}}^{-1}(t_n, x). \quad (4.7)$$

By (4.7) and Taylor's theorem, for each n , we get

$$\begin{aligned} C_n &= (\nabla u_{t_{n+1}}(x) - \nabla u_{t_n}(x))(x - Y_{t_{n+1}}(t_n, x)) \\ &= \nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n, x))(\nabla Y_{t_{n+1}}^{-1}(t_n, x) - I_d)(x - Y_{t_{n+1}}(t_n, x)) \\ &\quad + (Y_{t_{n+1}}^{-1}(t_n, x) - x)^* \tilde{\Theta}_n(x)(x - Y_{t_{n+1}}(t_n, x)) =: E_n + F_n, \end{aligned}$$

where

$$\tilde{\Theta}_n^{ij}(x) := \int_0^1 \partial_{ij} u_{t_n}(x + \theta(Y_{t_{n+1}}^{-1}(t_n, x) - x)) d\theta.$$

By Itô's formula, for each n , we have (see, also, Lemma 3.12 in [Kun04]),

$$\begin{aligned} \nabla Y_{t_{n+1}}(t_n, x)^{-1} &= I_d - \int_{]t_n, t_{n+1}]} \nabla Y_r(t_n, x)^{-1} \nabla \sigma_r^o(Y_r(t_n, x)) dw_r^o \\ &\quad + \int_{]t_n, t_{n+1}]} \nabla Y_r(t_n, x)^{-1} (\nabla \sigma_r^o(Y_r(t_n, y)) \nabla \sigma_r^o(Y_r(t_n, x)) - \nabla b_r(Y_r(t_n, x))) dr, \end{aligned}$$

and hence

$$\nabla Y_{t_{n+1}}^{-1}(t_n) - I_d = \nabla Y_{t_{n+1}}^{-1}(t_n, Y_{t_{n+1}}^{-1}(t_n, x)) - I_d =: G_{t_n, t_{n+1}}^{(1)}(Y_{t_{n+1}}^{-1}(t_n, x)) + G_{t_n, t_{n+1}}^{(2)}(Y_{t_{n+1}}^{-1}(t_n, x)),$$

where for $y \in \mathbf{R}^d$,

$$G_{t_n, t_{n+1}}^{(1)}(y) := \int_{]t_n, t_{n+1}]} \nabla Y_r(t_n, z)^{-1} (\nabla \sigma_r^o(Y_r(t_n, y)) \nabla \sigma_r^o(Y_r(t_n, y)) - \nabla b_r(Y_r(t_n, y))) dr$$

and

$$G_{t_n, t_{n+1}}^{(2)}(z) := - \int_{]t_n, t_{n+1}]} \nabla Y_r(t_n, y)^{-1} \nabla \sigma_r^o(Y_r(t_n, y)) dw_r^o.$$

By the Burkholder-Davis-Gundy inequality, Hölder's inequality, and the inequalities (3.2), (3.14), and (3.15), for each $p \geq 2$, there is a constant $N = N(p, d, N_0, T)$ such that for all x_1 and x_2 ,

$$\mathbf{E} \left[|G_{t_n, t_{n+1}}^{(2)}(x_1)|^p \right] \leq NM^{-p/2+1} \int_{]t_n, t_{n+1}]} \mathbf{E} \left[|\nabla Y_r(t_n, x_1)^{-1}|^p |\nabla \sigma_r(Y_r(t_n, x_1))|^p \right] dr \leq NM^{-p/2}$$

and

$$\begin{aligned} \mathbf{E} \left[|G_{t_n, t_{n+1}}^{(2)}(x_1) - G_{t_n, t_{n+1}}^{(2)}(x_2)|^p \right] &\leq NM^{-p/2+1} \int_{]t_n, t_{n+1}]} \mathbf{E} \left[|\nabla Y_r(t_n, x_1)^{-1} - \nabla Y_r(t_n, x_2)^{-1}|^p \right] dr \\ &\quad + NM^{-p/2+1} \int_{]t_n, t_{n+1}]} \left(\mathbf{E} \left[|\nabla Y_r(t_n, x_1)^{-1}|^{2p} \right] \right)^{1/2} \left(\mathbf{E} \left[|Y_r(t_n, x_1) - Y_r(t_n, x_2)|^{2p} \right] \right)^{1/2} dr \\ &\leq NM^{-p/2} |x - y|^p. \end{aligned}$$

Thus, by Corollary 5.3, we obtain that for all $p \geq 2$, $\epsilon > 0$, and $\delta < 1$, there is a constant $N = N(p, d, \delta, N_0, T)$ such that

$$\mathbf{E} \left[|r^{-\epsilon} G_{t_n, t_{n+1}}^{(2)}|^p \right] \leq NM^{-p/2}. \quad (4.8)$$

For each n , we have

$$\begin{aligned} E_n &= \nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n, x)) G_{t_n, t_{n+1}}^{(1)}(Y_{t_{n+1}}^{-1}(t_n, x))(x - Y_{t_{n+1}}(t_n, x)) \\ &\quad + \nabla u_{t_n}(x) G_{t_n, t_{n+1}}^{(2)}(x)(x - Y_{t_{n+1}}(t_n, x)) \\ &\quad + \nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n, x))(G_{t_n, t_{n+1}}^{(2)}(Y_{t_{n+1}}^{-1}(t_n, x)) - G_{t_n, t_{n+1}}^{(2)}(x))(x - Y_{t_{n+1}}(t_n, x)) \\ &\quad + (\nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n, x)) - \nabla u_{t_n}(x)) G_{t_n, t_{n+1}}^{(2)}(x)(x - Y_{t_{n+1}}(t_n, x)) \end{aligned}$$

One can easily check that

$$d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} \nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n, x)) G_{t_n, t_{n+1}}^{(1)}(Y_{t_{n+1}}^{-1}(t_n, x))(x - Y_{t_{n+1}}(t_n, x)) = 0. \quad (4.9)$$

Since $\nabla u_t(x)$ is jointly continuous in t and x and $Y_t^{-1}(s, x)$ is jointly in s and t , we have

$$d\mathbf{P} - \lim_{M \rightarrow \infty} \sup_n |\nabla u_{t_n^M}(Y_{t_{n+1}^M}^{-1}(t_n^M, x)) - \nabla u_{t_n^M}(x)| = 0.$$

Moreover, using Hölder's inequality, (4.8), and (3.1), we get

$$\sup_M \mathbf{E} \sum_{n=0}^{M-1} |G_{t_n, t_{n+1}}^{(2)}(x)| |x - Y_{t_{n+1}}(t_n, x)| < \infty,$$

and hence

$$d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} (\nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n, x)) - \nabla u_{t_n}(x)) G_{t_n, t_{n+1}}^{(2)}(x)(x - Y_{t_{n+1}}(t_n, x)) = 0. \quad (4.10)$$

We claim that

$$d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} \nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n, x)) (G_{t_n, t_{n+1}}^{(2)}(Y_{t_{n+1}}^{-1}(t_n, x)) - G_{t_n, t_{n+1}}^{(2)}(x))(x - Y_{t_{n+1}}(t_n, x)) = 0. \quad (4.11)$$

Set

$$J^M = \sum_{n=0}^{M-1} |G_{t_n, t_{n+1}}^{(2)}(Y_{t_{n+1}}^{-1}(t_n, x)) - G_{t_n, t_{n+1}}^{(2)}(x)| |x - Y_{t_{n+1}}(t_n, x)|.$$

For each $\bar{\delta}, \epsilon \in (0, 1)$, we have

$$\mathbf{P}(J^M > \bar{\delta}) \leq \mathbf{P}\left(J^M > \bar{\delta}, \max_n |Y_{t_{n+1}}^{-1}(t_n, x) - x| \leq \epsilon\right) + \mathbf{P}\left(\max_n |Y_{t_{n+1}}^{-1}(t_n, x) - x| > \epsilon\right).$$

By virtue of (4.8), there is a deterministic constant $N = N(x)$ independent of M such that for all $\omega \in V^M := \{\max_n |Y_{t_{n+1}}^{-1}(t_n, x) - x| \leq \epsilon\}$,

$$J^M \leq N\epsilon^\delta \sum_{n=0}^{M-1} [r_1^{-\epsilon} G_{t_n, t_{n+1}}^{(2)}]_\delta |x - Y_{t_{n+1}}(t_n, x)|,$$

which implies that

$$\mathbf{E} \mathbf{1}_{V^M} J^M \leq N\epsilon^\delta \mathbf{E} \sum_{n=0}^{M-1} \left([r_1^{-\epsilon} G_{t_n, t_{n+1}}^{(2)}]_\delta^2 + |x - Y_{t_{n+1}}(t_n, x)|^2 \right) \leq N\epsilon^\delta \sum_{n=0}^{M-1} M^{-1} \leq N\epsilon^\delta.$$

Applying Markov's inequality, we get

$$\mathbf{P}(J^M > \bar{\delta}, \max_n |Y_{t_{n+1}}^{-1}(t_n, x) - x| \leq \epsilon) \leq N \frac{\epsilon^\delta}{\bar{\delta}},$$

and hence for all $\bar{\delta} > 0$,

$$\lim_{M \rightarrow \infty} \mathbf{P}(J^M > \bar{\delta}) = 0,$$

which yields (4.11). Owing to (4.9), (4.10), and (4.11), we have

$$d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} E_n = \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} \nabla u_{t_n}(x) G_{t_n, t_{n+1}}^{(2)}(x) (x - Y_{t_{n+1}}(t_n, x)).$$

Proceeding as in the proof of part (1) of the claim, we obtain

$$\begin{aligned} & \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} K_n \\ &= \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} \nabla u_{t_n}(x) \int_{]t_n, t_{n+1}]} (\nabla Y_r(t_n, x)^{-1} - I_d) \nabla \sigma_r^{\mathcal{Q}}(x) dW_r^{\mathcal{Q}} \int_{]t_n, t_{n+1}]} \sigma_r^{\mathcal{Q}}(x) dW_r^{\mathcal{Q}} \\ & \quad + \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} \nabla u_{t_n}(x) \int_{]t_n, t_{n+1}]} \nabla \sigma_r^{\mathcal{Q}}(x) dW_r^{\mathcal{Q}} \int_{]t_n, t_{n+1}]} \sigma_r^{\mathcal{Q}}(x) dW_r^{\mathcal{Q}} \\ &= \int_{]s, t]} \sigma_r^{j\mathcal{Q}}(x) \partial_j \sigma_r^{i\mathcal{Q}}(x) \partial_i u_r(x) dr \end{aligned} \tag{4.12}$$

It is easy to check that for each n ,

$$\begin{aligned} F_n &= (Y_{t_{n+1}}^{-1}(t_n, x) - x)^* \tilde{\Theta}_n(x) (x - Y_{t_{n+1}}(t_n, x)) \\ &=: (G_{t_n, t_{n+1}}^{(3)}(Y_{t_{n+1}}^{-1}(t_n, x)) + G_{t_n, t_{n+1}}^{(4)}(Y_{t_{n+1}}^{-1}(t_n, x)))^* \tilde{\Theta}_n(x) (x - Y_{t_{n+1}}(t_n, x)), \end{aligned}$$

where for $y \in \mathbf{R}^d$,

$$G_{t_n, t_{n+1}}^{(3)}(y) := - \int_{]t_n, t_{n+1}]} b_r(Y_r(t_n, y)) dr, \quad G_{t_n, t_{n+1}}^{(4)}(y) := - \int_{]t_n, t_{n+1}]} \sigma_r^{\mathcal{Q}}(Y_r(t_n, y)) dW_r^{\mathcal{Q}}.$$

Arguing as in the proof of (4.12), we get

$$d\mathbf{P} - \lim_{M \rightarrow \infty} \sum_{n=0}^{M-1} F_n = \int_{[s,t]} \sigma_r^{i_0}(x) \sigma_r^{j_0}(x) \partial_{ij} u_r(x) dt,$$

which completes the proof of the claim. \square

By virtue of (4.5) and Claim 4.1, for all s and t with $s \leq t$ and x , \mathbf{P} -a.s.

$$u_t(x) = x + \int_{[s,t]} \left(\frac{1}{2} \sigma_r^{i_0}(x) \sigma_r^{j_0}(x) \partial_{ij} u_r(x) - \hat{b}_t^i(x) \partial_i u_r(x) \right) dr - \int_{[s,t]} \sigma_r^{i_0}(x) \partial_i u_r(x) dw_r^o. \quad (4.13)$$

Owing to Theorem 2.1, $u = u_t(x)$ has a modification that is jointly continuous in s and t and twice continuously differentiable in x . It is easy to check that the Lebesgue integral on the right-hand-side of (4.13) has a modification that is continuous in s , t , and x . Thus, the stochastic integral on the right-hand-side of (4.13) has a modification that is continuous in s , t , and x , and hence the equality in (4.13) holds \mathbf{P} -a.s. for all s and t with $s \leq t$ and x . This proves that $Y^{-1}(\tau) = Y_t^{-1}(\tau, x)$ solves (1.2). However, if $u^1(\tau), u^2(\tau) \in \mathfrak{C}_{cts}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d)$ are solutions of (1.2), then applying the Itô-Wentzell formula (see, e.g. Theorem 9 in Chapter 1, Section 4.8 in [Roz90]), we get that \mathbf{P} -a.s. for all t and x ,

$$u_t^1(\tau, Y_t(\tau, x)) = x = u_t^2(\tau, Y_t(\tau, x)),$$

which implies that \mathbf{P} -a.s. for all t and x , $u^1(\tau) = Y_t^{-1}(\tau, x) = u^2(\tau)$. Thus, $Y^{-1}(\tau) = Y_t^{-1}(\tau, x)$ is the unique solution of (1.2) in $\mathfrak{C}_{cts}^{\beta'}(\mathbf{R}^d; \mathbf{R}^d)$. \square

5 Appendix

Let V be an arbitrary Banach space. The following lemma and its corollaries are indispensable in this paper.

Lemma 5.1. *Let $Q \subseteq \mathbf{R}^d$ be an open bounded cube, $p \geq 1$, $\delta \in (0, 1]$, and f be a V -valued integrable function on Q such that*

$$[f]_{\delta;p;Q;V} := \left(\int_Q \int_Q \frac{|f(x) - f(y)|_V^p}{|x - y|^{2d+\delta p}} dx dy \right)^{1/p} < \infty.$$

Then f has a $C^\delta(Q; V)$ -modification and there is a constant $N = N(d, \delta, p)$ independent of f and Q such that

$$[f]_{\delta;Q;V} \leq N [f]_{\delta;p;Q;V}$$

and

$$\sup_{x \in Q} |f(x)|_V \leq N |Q|^{\delta/d} [f]_{\delta;p;Q;V} + |Q|^{-1/p} \left(\int_Q |f(x)|_V^p dx \right)^{1/p},$$

where $|Q|$ is the volume of the cube.

Proof. If $V = \mathbf{R}$, then the existence of a continuous modification of f and the estimate of $[f]_{\delta;Q}$ follows from Lemma 2 and Exercise 5 in Section 10.1 in [Kry08]. The proof for a general Banach space is the same. For all $x \in Q$, we have

$$\begin{aligned} |f(x)|_V &\leq \frac{1}{|Q|} \int_Q |f(x) - f(y)|_V dy + \frac{1}{|Q|} \int_Q |f(y)|_V dy \\ &\leq N \frac{1}{|Q|} [f]_{\delta,p;Q} \int_Q |x - y|^\delta dy + \frac{1}{|Q|} \int_Q |f(y)|_V dy \\ &\leq N |Q|^{\delta/d} [f]_{\delta,p;Q} + |Q|^{-1/p} \left(\int_Q |f(y)|_V^p dy \right)^{1/p}, \end{aligned}$$

which proves the second estimate. \square

The following is a direct consequence of Lemma 5.1.

Corollary 5.2. *Let $p \geq 1$, $\delta \in (0, 1]$, and f be a V -valued function on \mathbf{R}^d such that*

$$|f|_{\delta;p;V} := \left(\int_{\mathbf{R}^d} |f(x)|_V^p dx + \int_{|x-y|<1} \frac{|f(x) - f(y)|_V^p}{|x - y|^{2d+\delta p}} dx dy \right)^{1/p} < \infty.$$

Then f has a $C^\delta(\mathbf{R}^d; V)$ -modification and there is a constant $N = N(d, \delta, p)$ independent of f such that

$$|f|_{\delta;V} \leq N |f|_{\delta;p;V}.$$

Corollary 5.3. *Let X be a V -valued random field defined on \mathbf{R}^d . Assume that for some $p \geq 1$, $l \geq 0$, and $\beta \in (0, 1]$ with $\beta p > d$ there is a constant $\bar{N} > 0$ such that for all $x, y \in \mathbf{R}^d$,*

$$\mathbf{E} \left[|X(x)|_V^p \right] \leq \bar{N} r_1(x)^{lp} \quad (5.1)$$

and

$$\mathbf{E} \left[|X(x) - X(y)|_V^p \right] \leq \bar{N} [r_1(x)^{lp} + r_1(y)^{lp}] |x - y|^{\beta p}. \quad (5.2)$$

Then for any $\delta \in (0, \beta - \frac{d}{p})$ and $\epsilon > \frac{d}{p}$, there exists a $C^\delta(\mathbf{R}^d; V)$ -modification of $r_1^{-(l+\epsilon)} X$ and a constant $N = N(d, p, \delta, \epsilon)$ such that

$$\mathbf{E} \left[|r_1^{-(l+\epsilon)} X|_\delta^p \right] \leq N \bar{N}.$$

Proof. Fix $\delta \in (0, \beta - \frac{d}{p})$ and $\epsilon > \frac{d}{p}$. Owing to (5.1), there is a constant $N = N(d, p, \bar{N}, \delta, \epsilon)$ such that

$$\int_{\mathbf{R}^d} \mathbf{E} \left[|r_1(x)^{-(l+\epsilon)} X(x)|_V^p \right] dx \leq \bar{N} \int_{\mathbf{R}^d} r_1(x)^{-p\epsilon} dx \leq N \bar{N}.$$

By the mean value theorem, for each x and y and $\bar{p} \in \mathbf{R}$, we have

$$|r_1(x)^{\bar{p}} - r_1(y)^{\bar{p}}| \leq |\bar{p}| (r_1(x)^{\bar{p}-1} + r_1(y)^{\bar{p}-1}) |x - y|.$$

Appealing to (5.2) and (5), we obtain that there is a constant $N = N(d, p, \delta, \epsilon)$ such that

$$\begin{aligned} & \int_{|x-y|<1} \frac{\mathbf{E} \left[|r_1(x)^{-(l+\epsilon)} X(x) - r_1(y)^{-(l+\epsilon)} X(y)|_V^p \right]}{|x-y|^{2d+\delta p}} dx dy \\ & \leq \bar{N} \int_{|x-y|<1} \frac{r_1(x)^{-p\epsilon} + r_1(y)^{-p\epsilon}}{|x-y|^{2d+(\beta-\delta)p}} dx dy + \bar{N} \int_{|x-y|<1} \frac{r_1(y)^{pl} |r_1(x)^{-(l+\epsilon)} - r_1(y)^{-(l+\epsilon)}|^p}{|x-y|^{2d+\delta p}} dx dy \\ & \leq N\bar{N} + N\bar{N} \int_{|x-y|<1} \frac{r_1(x)^{-p(1+\epsilon)} + r_1(y)^{-p(1+\epsilon)}}{|x-y|^{2d-(1-\delta)p}} dx dy \leq N\bar{N}. \end{aligned}$$

Therefore, $\mathbf{E}[r_1^{-(l+\epsilon)} X]_{\delta,p}^p \leq N\bar{N}$, and hence, by Corollary 5.3, $r_1^{-(l+\epsilon)} X$ has a $C^\delta(\mathbf{R}^d; V)$ -modification and the estimate follows immediately. \square

Corollary 5.4. *Let $(X^{(n)})_{n \in \mathbf{N}}$ be a sequence of V -valued random field defined on \mathbf{R}^d . Assume that for some $p \geq 1$, $l \geq 0$ and $\beta \in (0, 1]$, with $\beta p > d$ there is a constant $\bar{N} > 0$ such that for all $x, y \in \mathbf{R}^d$ and $n \in \mathbf{N}$,*

$$\mathbf{E} \left[|X^{(n)}(x)|_V^p \right] \leq \bar{N} r_1(x)^{lp}$$

and

$$\mathbf{E} \left[|X^{(n)}(x) - X^{(n)}(y)|_V^p \right] \leq \bar{N} (r_1(x)^{lp} + r_1(y)^{lp}) |x - y|^{\beta p}.$$

Moreover, assume that for each $x \in \mathbf{R}^d$, $\lim_{n \rightarrow \infty} \mathbf{E} \left[|X^{(n)}(x)|^p \right] = 0$. Then for any $\delta \in (0, \beta - \frac{d}{p})$ and $\epsilon > \frac{d}{p}$,

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[|r_1^{-(l+\epsilon)} X^{(n)}|_\delta^p \right] = 0.$$

Proof. Fix $\delta \in (0, \beta - \frac{d}{p})$ and $\epsilon > \frac{d}{p}$. Using the Lebesgue dominated convergence theorem, we get

$$\lim_{n \rightarrow \infty} \int_{\mathbf{R}^d} \mathbf{E} \left[|r_1(x)^{-(l+\epsilon)} X^{(n)}(x)|_V^p \right] dx = 0,$$

and therefore for each $\zeta \in (0, 1)$,

$$\lim_n \int_{\zeta < |x-y| < 1} \frac{\mathbf{E} \left[|r_1(x)^{-(l+\epsilon)} X_n(x) - r_1(y)^{-(l+\epsilon)} X_n(y)|_V^p \right]}{|x-y|^{2d+\delta p}} dx dy = 0.$$

Repeating the proof of Corollary 5.3, we obtain that there is a constant N such that

$$\begin{aligned} & \int_{|x-y| \leq \zeta} \frac{\mathbf{E} \left[|r_1(x)^{-(l+\epsilon)} X^{(n)}(x) - r_1(y)^{-(l+\epsilon)} X^{(n)}(y)|_V^p \right]}{|x-y|^{2d+\delta p}} dx dy \\ & \leq \bar{N} \int_{|x-y| \leq \zeta} \frac{r_1(x)^{-p\epsilon} + r_1(y)^{-p\epsilon}}{|x-y|^{2d+(\delta-\beta)p}} dx dy + \bar{N} \int_{|x-y| \leq \zeta} \frac{r_1(x)^{-p(1+\epsilon)} + r_1(y)^{-p(1+\epsilon)}}{|x-y|^{2d+(\delta-1)p}} dx dy \\ & \leq \bar{N} \zeta^{\beta p - \delta p - d}. \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} \mathbf{E} \left[|r_1^{-(l+\epsilon)} X|_{\delta,p}^p \right] = 0$, and the statement follows. \square

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